A possible framework of the Lipkin model obeying the $SU(n)$ algebra in arbitrary fermion number. II: Two subalgebras in the $SU(n)$ Lipkin model and an approach to the construction of a linearly independent basis

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Based on the results for the minimum weight states obtained in the previous paper (I), an idea of how to construct the linearly independent basis is proposed for the $SU(n)$ Lipkin model. This idea starts in setting up m independent $SU(2)$ subalgebras in the cases with $n = 2m$ and $n = 2m + 1$ ($m = 2, 3, 4, \ldots$). The original representation is re-formed in terms of the spherical tensors for the $SU(n)$ generators built under the $SU(2)$ subalgebras. Through this re-formation, the $SU(m)$ subalgebra can be found. For constructing the linearly independent basis, not only the $SU(2)$ algebras but also the $SU(m)$ subalgebra play a central role. Some concrete results in the cases with $n = 2, 3, 4$, and 5 are presented.
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1. Introduction

This paper, (II), is a continuation of (I) [1] and we will mainly discuss an idea with the aid of which the linearly independent basis is constructed for the $SU(n)$ Lipkin model. As is well known, the $SU(n)$ Lipkin model is a classical model for describing certain phenomena observed in nuclei [2,3]. It is a kind of Lie algebraic model. Therefore, the first task in investigating this model is the construction of the minimum weight states. By the definition of the minimum weight states, this task may be performed independently of any Hamiltonian expressed as a function of the generators. In (I), we proposed an idea of how to construct the minimum weight states for the case with any total fermion number in n single-particle levels. A certain $SU(2)$ algebra, which we called the auxiliary $SU(2)$ algebra in (I), plays a central role for the minimum weight states. The raising operator in this auxiliary algebra is expressed in a form with the nth degree for the fermion creation operators and the Clifford numbers.

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After determining the minimum weight states, it may be natural to proceed to construct the linearly independent basis by applying various operators on the minimum weight states, such as the raising operator in the $SU(2)$ algebra. However, the following should be noted: Generally, it is impossible for determining the orthogonal sets to prepare a sufficient number of Hermitian operators that commute with one another. Therefore, in this paper, we will aim to construct a linearly independent basis, and then, for example, with the aid of the Schmidt method, we are able to obtain the orthogonal sets. In order to realize this aim, first, we pay attention to the point that, in the $SU(n)$ Lipkin model, we can define m independent $SU(2)$ subalgebras. Here, m is an integer related to n under the relation $n = 2m$ (n even) and $n = 2m + 1$ (n odd), where $m = 2, 3, \ldots$. Then, by adding these $SU(2)$ algebras, we get the $SU(2)$ algebra. With the use of this $SU(2)$ algebra, we can classify all the generators into scalars, vectors, and spinors. We can also prove that the set of scalars gives us the $SU(m)$ algebra. Therefore, the remaining vectors and spinors may be treated in the space given under the $SU(2)$ and $SU(m)$ algebra. The above may be nothing but the re-formation of the $SU(n)$ Lipkin model.

For a given value of the integer m ($m = 2, 3, \ldots$), we have two Lipkin models: the $SU(2m)$ and $SU(2m + 1)$ algebras. As a general argument, we know that the minimum weight states of the $SU(n)$ algebra are specified by $(n − 1)$ quantum numbers. In these numbers, m quantum numbers are related to m $SU(2)$ subalgebras, and the remaining ones are $(m − 1)$ for $n = 2m$ and m for $n = 2m + 1$, respectively. In these numbers, $(m − 1)$ quantum numbers may be related to the minimum weight states of the $SU(m)$ algebra. Therefore, in the case of the $SU(2m)$ algebra, the number $(n − 1)$ is decomposed into $m + (m − 1)$. In the case of the $SU(2m + 1)$ algebra, the number $(n − 1)$ is decomposed into $m + (m − 1) + 1$, where one quantum number is excess. We will show that this extra number is closely related to the spinor operators, because the spinors appear only in the case with $n = 2m + 1$. However, these features may be almost impossible to show in the general case; we will therefore discuss them through the cases with $n = 2, 3, 4$, and 5. The minimum weight states of these four cases were presented in (I) under the re-formed version. The cases with $n = 4$ and 5 correspond to $m = 2$ and they can be treated quite easily, because of $m = 2$, i.e., the $SU(2)$ algebra.

After giving m $SU(2)$ algebras in Sect. 2, the spherical tensor operators (scalar, vectors, and spinors) are defined. In Sect. 3, the Casimir operator as a quadratic form for the generators is diagonalized, and through this procedure the operators specifying the minimum weight states are introduced. Of course, the Hamiltonian given in (I) is rewritten in terms of the spherical tensors. In Sect. 4, our scheme for obtaining the linearly independent basis is presented. Following our scheme, in Sect. 5 the cases with $n = 2, 3, 4$, and 5 are treated. In Sect. 6, as a concluding remark, the connection between the $SU(2m)$ and $SU(2m + 1)$ Lipkin model is discussed. In the appendix, the $SU(m)$ subalgebra in the $SU(n)$ Lipkin model is presented in detail. Including the results given in (I), various formal aspects of the Lipkin model are also stressed in (II).

2. The $SU(n)$ generators of the Lipkin model in spherical tensor form

In Sect. 6 of Part I, we gave some examples of the re-formed generators of the Lipkin model. Following these examples, we will treat the present model under two cases: (i) n even, and (ii) n odd. In case (i), we introduce the integer m through

$$n = 2m; \quad n = 2, 4, \ldots, \quad \text{i.e.,} \quad m = 1, 2, \ldots. \tag{2.1}$$

We also treat case (ii) by the integer m, which is given by

$$n = 2m + 1; \quad n = 3, 5, \ldots, \quad \text{i.e.,} \quad m = 1, 2, \ldots. \tag{2.2}$$

In (I), we learned that the case with $m = 1$ corresponds to the $SU(2)$ and $SU(3)$ Lipkin model. The former is the $SU(2)$ algebra itself and the latter contains one $SU(2)$ subalgebra. On the other hand, the $SU(4)$ and $SU(5)$ Lipkin models corresponding to the case with $m = 2$ contain two $SU(2)$ subalgebras. Extending this aspect, we define m independent $SU(2)$ subalgebras. In the case with $n = 2m$, the following is given:

$$\widetilde{S}_+(m;r) = \widetilde{S}_{2m-2r}^{2m-2r+1}, \qquad \widetilde{S}_-(m;r) = \widetilde{S}_{2m-2r+1}^{2m-2r},$$

$$\widetilde{S}_0(m;r) = \frac{1}{2}\left(\widetilde{S}_{2m-2r+1}^{2m-2r+1} - \widetilde{S}_{2m-2r}^{2m-2r}\right) \qquad \text{for} \quad r = 1, 2, \ldots, m-1, \tag{2.3a}$$

$$\widetilde{S}_+(m;r) = \widetilde{S}^1, \qquad \widetilde{S}_-(m;r) = \widetilde{S}_1, \qquad \widetilde{S}_0(m;r) = \frac{1}{2}\widetilde{S}_1^1 \qquad \text{for} \quad r = m. \tag{2.3b}$$

The case with $n = 2m + 1$ gives us the form

$$\widetilde{S}_+(m;r) = \widetilde{S}_{2m-2r+1}^{2m-2r+2}, \qquad \widetilde{S}_-(m;r) = \widetilde{S}_{2m-2r+2}^{2m-2r+1},$$

$$\widetilde{S}_0(m;r) = \frac{1}{2}\left(\widetilde{S}_{2m-2r+2}^{2m-2r+2} - \widetilde{S}_{2m-2r+1}^{2m-2r+1}\right) \qquad \text{for} \quad r = 1, 2, \ldots, m-1, m. \tag{2.4}$$

We can see that the cases with $(m = 1, r = 1)$ and $(m = 2, r = 1, 2)$ reduce to the form treated in (I) for the cases with $n = 2, 3, 4$, and 5. Of course, we have the relation

$$[\,\widetilde{S}_+(m;r),\ \widetilde{S}_-(m;r')\,] = \delta_{rr'} \cdot \left(2\widetilde{S}_0(m;r)\right),$$

$$[\,\widetilde{S}_0(m;r),\ \widetilde{S}_\pm(m;r')\,] = \delta_{rr'} \cdot \left(\pm\widetilde{S}_\pm(m;r)\right). \tag{2.5}$$

In terms of total sum of $\widetilde{S}_{\pm,0}(m;r)$ for r, we obtain the $SU(2)$ algebra:

$$\widetilde{S}_{\pm,0}(m) = \sum_{r=1}^m \widetilde{S}_{\pm,0}(m;r). \tag{2.6}$$

In our re-formation, the $SU(2)$ algebra (2.6) will play a role in classifying all generators in terms of the spherical tensor representation.

The main task of this section is to express all the generators except the forms (2.3) and (2.4) in spherical tensor forms. For this task, it may be convenient to introduce two integers r and k:

(i) n even $(n = 2m)$, $r = 1, 2, \ldots, m-1$, $k = r+1, r+2, \ldots, m$, \hfill (2.7)

(ii) n odd $(n = 2m+1)$, $r = 1, 2, \ldots, m-1, m$, $k = r+1, r+2, \ldots, m$. \hfill (2.8)

In case (ii), if $r = m$, k becomes meaningless. In the case with $n = 2m$, $\widetilde{S}^p$ and $\widetilde{S}_q^p$ $(p > q)$ can be re-formed to scalars and vectors:

(α)scalars,

$$\widetilde{R}^{0,0}(m;r,k) = \begin{cases} \widetilde{S}_{2m-2k+1}^{2m-2r+1} + \widetilde{S}_{2m-2k}^{2m-2r} & (k = r+1, r+2, \ldots, m-1), \\[2mm] \widetilde{S}_1^{2m-2r+1} + \widetilde{S}^{2m-2r} & (k = m), \end{cases} \tag{2.9}$$

(β)vectors,

$$\widetilde{R}^{1,+1}(m;r,k) = \begin{cases} -\widetilde{S}_{2m-2k}^{2m-2r+1} & (k = r+1, r+2, \ldots, m-1), \\[2mm] -\widetilde{S}^{2m-2r+1} & (k = m), \end{cases} \tag{2.10}$$

$$\widetilde{R}^{1,0}(m;r,k) = \begin{cases} \dfrac{1}{\sqrt{2}} \left(\widetilde{S}^{2m-2r+1}_{2m-2k+1} - \widetilde{S}^{2m-2r}_{2m-2k} \right) & (k = r+1, r+2, \ldots, m-1), \\[12pt] \dfrac{1}{\sqrt{2}} \left(\widetilde{S}^{2m-2r+1}_{1} - \widetilde{S}^{2m-2r} \right) & (k = m), \end{cases} \tag{2.11}$$

$$\widetilde{R}^{1,-1}(m;r,k) = \begin{cases} \widetilde{S}^{2m-2r}_{2m-2k+1} & (k = r+1, r+2, \ldots, m-1), \\[10pt] \widetilde{S}^{2m-2r}_{1} & (k = m). \end{cases} \tag{2.12}$$

In the case with $n = 2m + 1$, spinors appear:

(γ) scalars,

$$\widetilde{R}^{0,0}(m;r,k) = \widetilde{S}^{2m-2r+2}_{2m-2k+2} + \widetilde{S}^{2m-2r+1}_{2m-2k+1} \quad \begin{pmatrix} r = 1, 2, \ldots, m-1 \\ k = r+1, r+2, \ldots, m \end{pmatrix}, \tag{2.13}$$

(δ) spinors,

$$\widetilde{R}^{\frac{1}{2},+\frac{1}{2}}(m;r) = \widetilde{S}^{2m-2r+2}, \qquad \widetilde{R}^{\frac{1}{2},-\frac{1}{2}}(m;r) = \widetilde{S}^{2m-2r+1} \quad (r = 1, 2, \ldots, m), \tag{2.14}$$

(ϵ) vectors,

$$\widetilde{R}^{1,+1}(m;r,k) = -\widetilde{S}^{2m-2r+2}_{2m-2k+1}, \tag{2.15}$$

$$\widetilde{R}^{1,0}(m;r,k) = \frac{1}{\sqrt{2}} \left(\widetilde{S}^{2m-2r+2}_{2m-2k+2} - \widetilde{S}^{2m-2r+1}_{2m-2k+1} \right), \tag{2.16}$$

$$\widetilde{R}^{1,-1}(m;r,k) = \widetilde{S}^{2m-2r+1}_{2m-2k+2} \quad \begin{pmatrix} r = 1, 2, \ldots, m-1, \\ k = r+1, r+2, \ldots, m \end{pmatrix}. \tag{2.17}$$

The forms (2.13)–(2.17) are also for $\widetilde{S}^p$ and $\widetilde{S}^p_q$ ($p > q$). The expressions (β) and (ϵ) give us scalars in the form

$$\widetilde{\Sigma}^{0,0}(m;r,k) = \sum_{\nu=\pm1,0} (-)^\nu \widetilde{R}^{1,\nu}(m;r,k) \widetilde{R}^{1,-\nu}(m;r,k). \tag{2.18}$$

The Hermitian conjugate of $\widetilde{R}^{\mu,\nu}(m;r,k)$ is denoted as

$$\left(\widetilde{R}^{\mu,\nu}(m;r,k) \right)^* = \widetilde{R}_{\mu,\nu}(m;r,k). \tag{2.19}$$

For now, the type $\widetilde{S}^p_p$ is re-formed as follows:

(i) $n = 2m$,

$$\widetilde{P}_0(m;r) = \begin{cases} \dfrac{1}{2} \left(\widetilde{S}^{2m-2r+1}_{2m-2r+1} + \widetilde{S}^{2m-2r}_{2m-2r} \right) & (r = 1, 2, \ldots, m-1), \\[12pt] \dfrac{1}{2} \widetilde{S}^1_1 & (r = m), \end{cases} \tag{2.20}$$

(ii) $n = 2m + 1$,

$$\widetilde{P}_0(m;r) = \frac{1}{2} \left(\widetilde{S}^{2m-2r+2}_{2m-2r+2} + \widetilde{S}^{2m-2r+1}_{2m-2r+1} \right) \quad (r = 1, 2, \ldots, m-1, m). \tag{2.21}$$

The operators $\widetilde{P}_0(m;r)$ should be compared with $\widetilde{S}_0(m;r)$. It is noted that we have the relation $\widetilde{P}_0(m;m) = \widetilde{S}_0(m;m)$ in case (i) and $\widetilde{P}_0(m;r)$ ($r = 1, 2, \ldots, m-1, m$) are not scalars, but in case (ii), $\widetilde{P}_0(m;r)$ ($r = 1, 2, \ldots, m-1, m$) are scalars. We will return to this point.

The total number of $\widetilde{S}^p(n)$ $(p = 1, 2, \ldots, n - 1)$ and $\widetilde{S}_q^p(n)$ $(p > q = 1, 2, \ldots, n - 2)$ is equal to

$$(n - 1) + \frac{(n^2 - 1) - (n - 1)}{2} = \frac{1}{2}n(n - 1). \tag{2.22}$$

On the other hand, the total number of $\widetilde{S}_+(m; r)$, $\widetilde{R}^{0,0}(m; r, k)$, and $\widetilde{R}^{1,\nu}(m; r, k)$ $(\nu = \pm 1, 0)$ in the case with $n = 2m$ is equal to

$$m + \sum_{r=1}^{m-1}(m - r) + 3\sum_{r=1}^{m-1}(m - r) = m(2m - 1) = \frac{1}{2}n(n - 1). \tag{2.23}$$

In the case with $n = 2m + 1$, the total number of $\widetilde{S}_+(m; r)$, $\widetilde{R}^{0,0}(m; r, k)$, $\widetilde{R}^{\frac{1}{2},\nu}(m; r)$ $(\nu = \pm 1/2)$, and $\widetilde{R}^{1,\nu}(m; r, k)$ $(\nu = \pm 1, 0)$ is equal to the relation (2.22):

$$m + \sum_{r=1}^{m-1}(m - r) + 2m + 3\sum_{r=1}^{m-1}(m - r) = m(2m + 1) = \frac{1}{2}n(n - 1). \tag{2.24}$$

The total number of $\widetilde{S}_p^p(n)$ $(p = 1, 2, \ldots, n - 1)$ is equal to $(n - 1)$. In the case with $n = 2m$, the total number of $\widetilde{S}_0(m; r)$ and $\widetilde{P}_0(m; r)$ is equal to

$$m + m = 2m = n. \tag{2.25}$$

We have the difference $n - (n - 1) = 1$. The reason why we have this difference comes from the double counting for $(1/2)\widetilde{S}_1^1$. Later, we will show that this double counting does not give us any trouble. In the case with $n = 2m + 1$, the total number of $\widetilde{S}_0(m; r)$ and $\widetilde{P}_0(m; r)$ is equal to

$$m + m = 2m = n - 1. \tag{2.26}$$

Thus, we re-formed the $SU(n)$ generators in the Lipkin model in the form of the spherical tensor representation for the $SU(2)$ algebra $(\widetilde{S}_{\pm,0}(m))$.

Finally, we give a comment. We presented a possible re-formation of the original representation of the $SU(n)$ Lipkin model. Concerning the $SU(2)$ subalgebra, it starts in the expressions (2.3) and (2.4). However, it must be noted that these forms are not unique. If the indices $i = 1, 2, \ldots, n - 1$ are changed from the forms (2.3) and (2.4), we obtain various forms. For example, in the case with $n = 2m = 4$, $\widetilde{S}_{\pm,0}(m; r)$ in the relation (2.3) can be expressed as

$$\widetilde{S}_+(2; 1) = \widetilde{S}_2^3, \quad \widetilde{S}_-(2; 1) = \widetilde{S}_3^2, \quad \widetilde{S}_0(2; 1) = \frac{1}{2}\left(\widetilde{S}_3^3 - \widetilde{S}_2^2\right), \tag{2.27a}$$

$$\widetilde{S}_+(2; 2) = \widetilde{S}^1, \quad \widetilde{S}_-(2; 2) = \widetilde{S}_1, \quad \widetilde{S}_0(2; 2) = \frac{1}{2}\widetilde{S}_1^1. \tag{2.27b}$$

For the form (2.27), we exchange the indices $i = 1$ and 2, and then they become

$$\widetilde{S}_+(2; 1) = \widetilde{S}_1^3, \quad \widetilde{S}_-(2; 1) = \widetilde{S}_3^1, \quad \widetilde{S}_0(2; 1) = \frac{1}{2}\left(\widetilde{S}_3^3 - \widetilde{S}_1^1\right), \tag{2.28a}$$

$$\widetilde{S}_+(2; 2) = \widetilde{S}^2, \quad \widetilde{S}_-(2; 2) = \widetilde{S}_2, \quad \widetilde{S}_0(2; 2) = \frac{1}{2}\widetilde{S}_2^2. \tag{2.28b}$$

The form (2.28) also obeys the $SU(2)$ algebra.

3. The Casimir operator in the present representation and its diagonalization

In this section, we will discuss the Casimir operator, $\widetilde{\Gamma}_{SU(n)}$, shown in (I), relation (2.4):

$$\widetilde{\Gamma}_{SU(n)} = \frac{1}{2}\left[\sum_{p=1}^{n-1}\left[\,\widetilde{S}^p,\,\widetilde{S}_p\,\right]_+ + \sum_{p=2}^{n-1}\sum_{q=1}^{p-1}\left[\,\widetilde{S}_q^p,\,\widetilde{S}_p^q\,\right]_+ + \sum_{p=1}^{n-1}\left(\widetilde{S}_p^p\right)^2 - \frac{1}{n}\left(\sum_{p=1}^{n-1}\widetilde{S}_p^p\right)^2\right]. \quad (3.1)$$

Here, $[\,\widetilde{A},\,\widetilde{B}\,]_+ = \widetilde{A}\widetilde{B} + \widetilde{B}\widetilde{A}$. We can see in the form (3.1) that $\widetilde{\Gamma}_{SU(n)}$, which is of quadratic form with respect to the $SU(n)$ generators, is not diagonal. Then, we will consider the diagonalization. The operator $\widetilde{\Gamma}_{SU(n)}$ can be rewritten as

$$\widetilde{\Gamma}_{SU(n)} = \sum_{r=1}^{m}\widetilde{\boldsymbol{S}}(m;r)^2 + \widetilde{\boldsymbol{\Gamma}}_m + \widetilde{\boldsymbol{\Gamma}}_m^{(2)}, \quad (3.2)$$

$$\widetilde{\boldsymbol{S}}(m;r)^2 = \frac{1}{2}\left[\,\widetilde{S}_+(m;r),\,\widetilde{S}_-(m;r)\,\right]_+ + \widetilde{S}_0(m;r)^2. \quad (3.3)$$

The part $\widetilde{\boldsymbol{S}}(m;r)^2$ denotes the Casimir operator of the $SU(2)$ subalgebra $(\widetilde{S}_{\pm,0}(m;r))$, and $\widetilde{\boldsymbol{\Gamma}}_m$ is given separately in the cases with $n = 2m$ and $2m+1$:

(i) $n = 2m$,

$$\widetilde{\boldsymbol{\Gamma}}_m = \frac{1}{2}\sum_{r=1}^{m-1}\sum_{k=r+1}^{m}\left(\left[\,\widetilde{R}^{0,0}(m;r,k),\,\widetilde{R}_{0,0}(m;r,k)\,\right]_+\right.$$
$$\left. + \sum_{\nu=\pm 1,0}\left[\,\widetilde{R}^{1,\nu}(m;r,k),\,\widetilde{R}_{1,\nu}(m;r,k)\,\right]_+\right), \quad (3.4\mathrm{a})$$

(ii) $n = 2m+1$,

$$\widetilde{\boldsymbol{\Gamma}}_m = \frac{1}{2}\sum_{r=1}^{m-1}\sum_{k=r+1}^{m}\left(\left[\,\widetilde{R}^{0,0}(m;r,k),\,\widetilde{R}_{0,0}(m;r,k)\,\right]_+\right.$$
$$\left. + \sum_{\nu=\pm 1,0}\left[\,\widetilde{R}^{1,\nu}(m;r,k),\,\widetilde{R}_{1,\nu}(m;r,k)\,\right]_+\right)$$
$$+ \frac{1}{2}\sum_{r=1}^{m}\sum_{\nu=\pm 1/2}\left[\,\widetilde{R}^{\frac{1}{2},\nu}(m;r),\,\widetilde{R}_{\frac{1}{2},\nu}(m;r)\,\right]_+. \quad (3.4\mathrm{b})$$

The part $\widetilde{\boldsymbol{\Gamma}}_m^{(2)}$ is expressed as

$$\widetilde{\boldsymbol{\Gamma}}_m^{(2)} = \sum_{r=1}^{m}\left(\widetilde{P}_0(m;r)\right)^2 - \frac{2}{n}\left(\sum_{r=1}^{m}\widetilde{P}_0(m;r)\right)^2. \quad (3.5)$$

The operators $\widetilde{\boldsymbol{\Gamma}}_m$ and $\widetilde{\boldsymbol{\Gamma}}_m^{(2)}$ are quadratic. However, $\widetilde{\boldsymbol{\Gamma}}_m$ is diagonalized, but $\widetilde{\boldsymbol{\Gamma}}_m^{(2)}$ is not diagonalized. The above is our re-formation of $\widetilde{\Gamma}_{SU(n)}$.

The operator $\widetilde{\boldsymbol{\Gamma}}_m^{(2)}$ can be diagonalized as follows:

$$\widetilde{\boldsymbol{\Gamma}}_m^{(2)} = \sum_{r=1}^{m-1}\left(\widetilde{Q}_0(m;r)\right)^2 + \left(\frac{n-2m}{n}\right)\left(\widetilde{Q}_0(m;m)\right)^2, \quad (3.6)$$

i.e.,

(i) $\quad n = 2m, \qquad \widetilde{\mathbf{\Gamma}}^{(2)}_{m=1} = 0, \qquad \widetilde{\mathbf{\Gamma}}^{(2)}_{m \geq 2} = \sum_{r=1}^{m-1} \left(\widetilde{Q}_0(m; r) \right)^2,$ $\qquad$ (3.7a)

(ii) $\quad n = 2m + 1, \quad \widetilde{\mathbf{\Gamma}}^{(2)}_{m=1} = \frac{1}{3} \left(\widetilde{Q}_0(1; 1) \right)^2,$

$$\widetilde{\mathbf{\Gamma}}^{(2)}_{m \geq 2} = \sum_{r=1}^{m-1} \left(\widetilde{Q}_0(m; r) \right)^2 + \frac{1}{2m + 1} \left(\widetilde{Q}_0(m; m) \right)^2.$$ $\qquad$ (3.7b)

It should be noted that the second term on the right-hand side of the relation (3.6) disappears in the case with $n = 2m$. The operator $\widetilde{Q}_0(m; r)$ is obtained from $\widetilde{P}_0(m; r)$ by the orthogonal transformation

$$\widetilde{Q}_0(m; r) = \sum_{r'=1}^{m} C_{r,r'}(m) \widetilde{P}_0(m; r'),$$ $\qquad$ (3.8a)

conversely,

$$\widetilde{P}_0(m; r) = \sum_{r'=1}^{m} C_{r',r}(m) \widetilde{Q}_0(m; r').$$ $\qquad$ (3.8b)

Here, $(C_{r',r}(m))$ denotes an orthogonal matrix:

$$C_{r'=m,r}(m) = \frac{1}{\sqrt{m}}, \qquad \sum_{r''=1}^{m} C_{r,r''}(m) C_{r',r''}(m) = \sum_{r''=1}^{m} C_{r'',r}(m) C_{r'',r'}(m) = \delta_{r,r'}.$$ $\qquad$ (3.9)

In this paper, we will adopt the following form for $C_{r',r}(m)$:

$$C_{r',r}(m) = \left\langle \frac{m-1}{2}, r - \frac{m+1}{2}, \frac{m-1}{2}, -\left(r - \frac{m+1}{2} \right) \middle| m - r', 0 \right\rangle (-)^{r-r'}.$$ $\qquad$ (3.10)

Here, $C_{r',r}(m)$ except for the phase factor $(-)^{r-r'}$ denotes the Clebsch–Gordan coefficient of type $\langle j_1, m_1, j_1, -m | j_2, 0 \rangle$, and it is given as

$$C_{r',r}(m) = (-)^{m-r'} \left(\sqrt{D_{m,r'}} \right)^{-1} E_{r',r}(m),$$ $\qquad$ (3.10a)

$$D_{m,r'} = \frac{\prod_{\mu=0}^{m-r'} (m + \mu)(m - \mu)}{m(2(m - r') + 1)},$$ $\qquad$ (3.10b)

$$E_{r',r}(m) = \sum_{s} (-)^s \frac{(r - 1 + s)!(2m - r' - r - s)!}{s!(m - r - s)!(m - r' - s)!(r' + r - m - 1 + s)!},$$ $\qquad$ (3.10c)

$$F_{r',r}(m) = (-)^{m-r'} \left(D_{m,r'} \right)^{-1} E_{r',r}(m).$$ $\qquad$ (3.10d)

Later, $F_{r',r}(m)$ will be used. For example, we have

$$C_{r'=m,r}(m) = \frac{1}{\sqrt{m}}, \qquad D_{m,r'=m} = m, \qquad E_{r'=m,r}(m) = 1,$$ $\qquad$ (3.11a)

$$C_{r'=m-1,r}(m) = -\sqrt{\frac{3}{m(m+1)(m-1)}} (2r - (m + 1)),$$

$$D_{m,r'=m-1} = \frac{m(m+1)(m-1)}{3}, \qquad E_{r'=m-1,r}(m) = 2r - (m+1). \qquad (3.11b)$$

The relations (3.8) and (3.9) lead us to the expression (3.6). We note that $\widetilde{Q}_0(m;r)$ for $r = 1, 2, \ldots, m-1$ are scalar in the case with $n = 2m$ (this will be shown later). In the case with $n = 2m$, $\widetilde{Q}_0(m;m)$ disappears in $\widetilde{\mathbf{\Gamma}}_m^{(2)}$, because of the factor $(n - 2m)$ in the relation (3.6). In this case, in spite of the relation $\widetilde{P}_0(m;m) = \widetilde{S}_0(m;m)$, we have treated these two as if they are independent of each other. As was already mentioned in the relation (2.25), the total number of $\widetilde{P}_0(m;r)$, i.e., $\widetilde{Q}_0(m;r)$, becomes equal to that of $\widetilde{S}_0(m;r)$. Originally, the total number of $\widetilde{S}_p^p(n)$ is equal to $n - 1$, i.e., $2m - 1$, and in this number the integer m is fixed as the total number of $\widetilde{S}_0(m;r)$. Therefore, the total number of $\widetilde{Q}_0(m;r)$ should be reduced to $m - 1$. We can see in the relation (3.7a) that $\widetilde{Q}_0(m;m)$ disappears; in other words, only $\widetilde{Q}_0(m;r)$ $(r = 1, 2, \ldots, m-1)$ appear. The case with $n = 2m + 1$ does not contain such a problem, because of $n = 2m + 1$, i.e., $n - 1 = 2m$.

Now, let us derive the formulae for obtaining the eigenvalues of $\widetilde{\Gamma}_{SU(n)}$. To do this, we must rewrite $(1/2) \cdot [\, \widetilde{S}_+(m;r),\ \widetilde{S}_-(m;r)\,]_+$ and other terms of the anti-commutators to the forms $\widetilde{S}_+(m;r)\widetilde{S}_-(m;r)$ and others. As a result, the linear terms for the generators appear. The part $\sum_{r=1}^m \widetilde{S}(m;r)^2$ shown in the relation (3.3) is simple:

$$\text{the linear term} = -\sum_{r=1}^m \widetilde{S}_0(m;r). \qquad (3.12)$$

The linear terms coming from the relation (3.4), which we denote as $\widetilde{\mathbf{\Gamma}}_m^{(1)}$, can be expressed in the form

$$\widetilde{\mathbf{\Gamma}}_m^{(1)} = 2\sum_{r=1}^m \left(2r - \left(\frac{n}{2} + 1\right)\right) \widetilde{P}_0(m;r)$$

$$= 2\sum_{r=1}^m (2r - (m+1)) \widetilde{P}_0(m;r) - (n - 2m)\sum_{r=1}^m \widetilde{P}_0(m;r). \qquad (3.13)$$

With the use of the relations (3.8) and (3.11), $\widetilde{\mathbf{\Gamma}}_m^{(1)}$ can be expressed as

$$\widetilde{\mathbf{\Gamma}}_m^{(1)} = -2\sqrt{D_{m,m-1}}\widetilde{Q}_0(m, m-1) - (n - 2m)\sqrt{m}\widetilde{Q}_0(m;m). \qquad (3.14)$$

We can see that in the case with $n = 2m$, $\widetilde{Q}_0(m;m)$ also disappears in $\widetilde{\mathbf{\Gamma}}_m^{(1)}$.

The sum $\widetilde{\mathbf{\Gamma}}_m^{(2)} + \widetilde{\mathbf{\Gamma}}_m^{(1)}$ $(= \widetilde{\mathbf{\Gamma}}_m^{(0)})$ can be expressed in the following form:

$$\widetilde{\mathbf{\Gamma}}_m^{(0)} = \sum_{r=1}^{m-1} \widetilde{Q}_0(m;r) \left(\widetilde{Q}_0(m;r) - 2\sqrt{D_{m,r}}\delta_{r,m-1}\right)$$

$$+ \left(\frac{n - 2m}{n}\right) \widetilde{Q}_0(m;m) \left(\widetilde{Q}_0(m;m) - n\sqrt{m}\right). \qquad (3.15)$$

By changing the scale of $\widetilde{Q}_0(m;r)$, we can eliminate the irrational numbers contained in $\widetilde{Q}_0(m;r)$:

$$\widetilde{\mathbf{\Gamma}}_m^{(0)} = \frac{1}{4}\sum_{r=1}^{m-1} D_{m,r}\widetilde{R}_0(m;r) \left(\widetilde{R}_0(m;r) - 4\delta_{r,m-1}\right)$$

$$+ (n - 2m) \cdot \frac{1}{nm}\widetilde{R}_0(m;m) \left(\widetilde{R}_0(m;m) - nm\right), \qquad (3.16)$$

$$\widetilde{R}_0(m;r) = 2\left(\sqrt{D_{m,r}}\right)^{-1}\widetilde{Q}_0(m;r)$$

$$= 2\sum_{r'=1}^{m} F_{r,r'}(m)\widetilde{P}_0(m;r') \quad (r = 1, 2, \ldots, m-1), \tag{3.17a}$$

$$\widetilde{R}_0(m;m) = \sqrt{m}\widetilde{Q}_0(m;m) = \sum_{r'=1}^{m}\widetilde{P}_0(m;r'). \tag{3.17b}$$

The linear terms indicate the quantum fluctuations. We see that the fluctuations appear only at the points $r = m$ and $m-1$. If the orthogonal matrix (3.10) is not adopted, they may spread over various points of r. In the case with $n = 2m$, $\widetilde{P}_0(m;r)$ $(r = 1, 2, \ldots, m-1, m)$ is not scalar. But, it can be easily proved that $\widetilde{Q}_0(m;r)$, i.e., $\widetilde{R}_0(m;r)$ $(r = 1, 2, \ldots, m-1)$ is scalar, and we see that, in $\widetilde{\boldsymbol{\Gamma}}_m^{(0)}$, $\widetilde{R}_0(m;m)$ in the case with $n = 2m$ disappears. Using the results in (I), we can calculate the eigenvalues of $\widetilde{R}_0(m;r)$ for a given minimum weight state.

Let us show that $\widetilde{Q}_0(m;r)$ for $r = 1, 2, \ldots, m-1$ in the case with $n = 2m$ is scalar. For example, we have

$$[\,\widetilde{S}_+(m),\,\widetilde{P}_0(m;r)\,] = -\widetilde{S}^1 \qquad (r = 1, 2, \ldots, m-1, m). \tag{3.18}$$

Then, the relation (3.18) with (3.9) and (3.11a) lead us to

$$[\,\widetilde{S}_+(m),\,\widetilde{Q}_0(m;r)\,] = \sum_{r'=1}^{m} C_{r,r'}(m)[\,\widetilde{S}_+(m),\,\widetilde{P}_0(m)\,]$$

$$= -\left(\sum_{r'=1}^{m} C_{r,r'}(m)\right)\widetilde{S}^1 = -\delta_{r,m}\sqrt{m}\,\widetilde{S}^1. \tag{3.19}$$

In the case with $n = 2m+1$ for $r = 1, 2, \ldots, m-1, m$, we have

$$[\,\widetilde{S}_+(m),\,\widetilde{P}_0(m;r)\,] = 0, \quad \text{i.e.,} \quad [\,\widetilde{S}_+(m),\,\widetilde{Q}_0(m;r)\,] = 0. \tag{3.20}$$

Other cases with $\widetilde{S}_-(m)$ and $\widetilde{S}_0(m)$ are also in the same situation as the above.

Further, for $r = 1, 2, \ldots, m-1$ in the case with $n = 2m$, the following relation is derived:

$$\widetilde{Q}_0(m;r) = \sum_{r'=1}^{m-1} C_{r,r'}(m)\widetilde{\mathcal{P}}_0(m;r'), \tag{3.21a}$$

conversely,

$$\widetilde{\mathcal{P}}_0(m;r) - \frac{1}{m}\sum_{r'=1}^{m-1}\widetilde{\mathcal{P}}_0(m;r') = \sum_{r'=1}^{m-1} C_{r',r}(m)\widetilde{Q}_0(m;r'). \tag{3.21b}$$

Here, $\widetilde{\mathcal{P}}_0(m;r)$ is defined as

$$\widetilde{\mathcal{P}}_0(m;r) = \widetilde{P}_0(m;r) - \widetilde{P}_0(m;m) = \frac{1}{2}\left(\widetilde{S}_{2m-2r+1}^{2m-2r+1} + \widetilde{S}_{2m-2r}^{2m-2r} - \widetilde{S}_1^1\right). \tag{3.22}$$

Therefore, in place of using $\widetilde{Q}_0(m;r)$, it may be permitted to also use $\widetilde{\mathcal{P}}_0(m;r)$. In parallel to the case with $n = 2m$, we also use $\widetilde{\mathcal{P}}_0(m;r)$ in the case with $n = 2m+1$:

$$\widetilde{\mathcal{P}}_0(m;r) = \widetilde{P}_0(m;r) = \frac{1}{2}\left(\widetilde{S}_{2m-2r+2}^{2m-2r+2} + \widetilde{S}_{2m-2r+1}^{2m-2r+1}\right) \quad (r = 1, 2, \ldots, m-1, m). \tag{3.23}$$

However, the relation (3.21) does not hold in the case with $n = 2m + 1$.

In (I), we showed an example of the Hamiltonian:

$$\widetilde{H}(n) = \widetilde{H}_0(n) + \widetilde{H}_1(n). \tag{3.24}$$

The term $\widetilde{H}_0(n)$ is the Hamiltonian of individual levels with energies ϵ_p:

$$\widetilde{H}_0(n) = \sum_{p=0}^{n-1} \epsilon_p \widetilde{N}_p(n) = \sum_{p=1}^{n-1} \epsilon_p \widetilde{S}_p^p(n), \tag{3.25a}$$

$$\sum_{p=0}^{n-1} \epsilon_p = 0, \quad \epsilon_0 \leq \epsilon_1 \leq \cdots \leq \epsilon_{n-1}. \tag{3.25b}$$

The part $\widetilde{H}_1(n)$ is the interaction term:

$$\widetilde{H}_1(n) = -G \sum_{p=1}^{n-1} \left[\left(\widetilde{S}^p(n) \right)^2 + \left(\widetilde{S}_p(n) \right)^2 \right] \quad (G : \text{coupling constant}). \tag{3.26}$$

We can rewrite $\widetilde{H}(n)$ in our new representation. The part $\widetilde{H}_0(n)$ can be rewritten in the form given below. In the case with $n = 2m$, the relations (2.3) and (2.26) give us

$$\widetilde{S}_{2m-2r+1}^{2m-2r+1} = \widetilde{P}_0(m;r) + \widetilde{S}_0(m;r), \quad \widetilde{S}_{2m-2r}^{2m-2r} = \widetilde{P}_0(m;r) - \widetilde{S}_0(m;r)$$
$$(r = 1, 2, \ldots, m-1), \tag{3.27a}$$

$$\widetilde{S}_1^1 = \widetilde{P}_0(m;m) + \widetilde{S}_0(m;m) \quad \left(r = m, \widetilde{P}_0(m;m) = \widetilde{S}_0(m;m) \right). \tag{3.27b}$$

Then, $\widetilde{H}_0(n)$ can be expressed in the form

$$\widetilde{H}_0(n) = \sum_{r=1}^{m} (\epsilon_{2m-2r+1} + \epsilon_{2m-2r}) \widetilde{P}_0(m;r) + \sum_{r=1}^{m} (\epsilon_{2m-2r+1} - \epsilon_{2m-2r}) \widetilde{S}_0(m;r)$$
$$= \sum_{r=1}^{m-1} (\epsilon_{2m-2r+1} + \epsilon_{2m-2r}) \widetilde{P}_0(m;r) + \sum_{r=1}^{m} (\epsilon_{2m-2r+1} - \epsilon_{2m-2r}) \widetilde{S}_0(m;r). \tag{3.28}$$

For the above rewriting, the relation (3.25b) is used. In the case with $n = 2m + 1$, we have

$$\widetilde{S}_{2m-2r+2}^{2m-2r+2} = \widetilde{P}_0(m;r) + \widetilde{S}_0(m;r), \quad \widetilde{S}_{2m-2r+1}^{2m-2r+1} = \widetilde{P}_0(m;r) - \widetilde{S}_0(m;r)$$
$$(r = 1, 2, \ldots, m-1, m). \tag{3.29}$$

Then, we obtain

$$\widetilde{H}_0(n) = \sum_{r=1}^{m} (\epsilon_{2m-2r+2} + \epsilon_{2m-2r+1}) \widetilde{P}_0(m;r) + \sum_{r=1}^{m} (\epsilon_{2m-2r+2} - \epsilon_{2m-2r+1}) \widetilde{S}_0(m;r). \tag{3.30}$$

The term $\widetilde{H}_1(n)$ is rather simple. In the case with $n = 2m$, we have

$$\widetilde{S}^{2m-2r+1} = -\widetilde{R}^{1,1}(m;r,m),$$

$$\widetilde{S}^{2m-2r} = \frac{1}{\sqrt{2}} \left(\widetilde{R}^{0,0}(m;r,m) - \widetilde{R}^{1,0}(m;r,m) \right) \quad (r = 1, 2, \ldots, m-1),$$

$$\widetilde{S}_+(m;m) = \widetilde{S}^1. \tag{3.31}$$

In the case with $n = 2m + 1$, we obtain

$$\widetilde{S}^{2m-2r+2} = \widetilde{R}^{\frac{1}{2},\frac{1}{2}}(m;r),$$

$$\widetilde{S}^{2m-2r+1} = \widetilde{R}^{\frac{1}{2},-\frac{1}{2}}(m;r) \quad (r = 1, 2, \ldots, m-1, m). \tag{3.32}$$

Then, we have

$$\sum_{p=1}^{n-1} \left(\widetilde{S}^p\right)^2 = \sum_{r=1}^{m-1} \left[\left(\widetilde{R}^{1,1}(m;r,m)\right)^2 + \frac{1}{2}\left(\widetilde{R}^{0,0}(m;r,m) - \widetilde{R}^{1,0}(m;r,m)\right)^2\right]$$

$$+ \left(\widetilde{S}_+(m;m)\right)^2 \qquad \text{(the case with } n = 2m\text{)}, \tag{3.33}$$

$$\sum_{p=1}^{n-1} \left(\widetilde{S}^p\right)^2 = \sum_{r=1}^{m} \left[\left(\widetilde{R}^{\frac{1}{2},\frac{1}{2}}(m;r)\right)^2 + \left(\widetilde{R}^{\frac{1}{2},-\frac{1}{2}}(m;r)\right)^2\right]$$

$$\text{(the case with } n = 2m + 1\text{)}. \tag{3.34}$$

The part $\widetilde{H}_1(n)$ can be given by substituting the relations (3.33) and (3.34) and their Hermitian conjugate into the relation (3.26).

4. A possible scheme for constructing the linearly independent basis

In (I), we investigated the structure of the minimum weight states of the $SU(n)$ Lipkin model. In Sect. 6 of (I), we sketched the minimum weight states in the representation developed in the present paper (II) for the cases with $n = 2, 3, 4$, and 5. Although our final aim of this section is to present our idea for constructing the linearly independent basis, the first task is to generalize the above four cases to the case with arbitrary n.

Let the eigenvalues of $\widetilde{S}_0(m;r)$ and $\widetilde{R}_0(m;r)$ for the minimum weight state be denoted as $-\sigma^r$ and $-\rho^r$, respectively. Here, $\widetilde{S}_0(m;r)$ is given in the relation (2.3) for the case with $n = 2m$ and (2.4) for the case with $n = 2m + 1$:

(i) $n = 2m$,

$$\widetilde{S}_0(m;r) = \begin{cases} \dfrac{1}{2}\left(\widetilde{S}^{2m-2r+1}_{2m-2r+1} - \widetilde{S}^{2m-2r}_{2m-2r}\right) & (r = 1, 2, \ldots, m-1), \\[2ex] \dfrac{1}{2}\widetilde{S}^1_1 & (r = m), \end{cases} \tag{4.1a}$$

(ii) $n = 2m + 1$,

$$\widetilde{S}_0(m;r) = \frac{1}{2}\left(\widetilde{S}^{2m-2r+2}_{2m-2r+2} - \widetilde{S}^{2m-2r+1}_{2m-2r+1}\right) \quad (r = 1, 2, \ldots, m-1, m). \tag{4.1b}$$

On the other hand, the relations (3.17) and (3.21) with (3.10) give us the operators $\widetilde{R}_0(m;r)$:

(i) $n = 2m$,

$$\widetilde{R}_0(m;r) = 2\sum_{r'=1}^{m-1} F_{r,r'}(m) \cdot \widetilde{\mathcal{P}}_0(m;r') \qquad (r = 1, 2, \ldots, m-1), \tag{4.2a}$$

(ii) $n = 2m + 1$,

$$
\widetilde{R}_0(m; r) = \begin{cases} 2 \displaystyle\sum_{r'=1}^{m} F_{r,r'}(m) \widetilde{\mathcal{P}}_0(m; r') & (r = 1, 2, \ldots, m-1), \\[2ex] \displaystyle\sum_{r'=1}^{m} \widetilde{\mathcal{P}}_0(m; r') & (r = m). \end{cases} \tag{4.2b}
$$

Here, $\widetilde{\mathcal{P}}_0(m; r)$ is shown in the relations (3.22) and (3.23). We notice that there exists a difference between the case with $n = 2m$ and with $n = 2m + 1$. For σ^r, the superscript r is in the range $r = 1, 2, \ldots, m-1, m$, for both cases. However, for ρ^r, the range of r in the case with $n = 2m$ is $r = 1, 2, \ldots, m-1$, and in the case with $n = 2m+1$, $r = 1, 2, \ldots, m-1, m$. The reason has already been mentioned, and is also given in the appendix. Then, the minimum weight state $|\min(n)\rangle$ can be expressed formally in the form

$$
|\min(n)\rangle = \begin{cases} |N; \vec{\rho}, \vec{\sigma}\rangle & (n = 2m), \\ |N; \vec{\rho}, \rho_m, \vec{\sigma}\rangle & (n = 2m + 1). \end{cases} \tag{4.3}
$$

Here, we adopt the abbreviations

$$
\vec{\rho} = \rho^1, \rho^2, \ldots, \rho^{m-1}, \qquad \vec{\sigma} = \sigma^1, \sigma^2, \ldots, \sigma^{m-1}, \sigma^m. \tag{4.4}
$$

As is clear from the relation (4.2), it may be enough to obtain the eigenvalues of $\widetilde{\mathcal{P}}_0(m; r)$ ($r = 1, 2, \ldots, m-1$ or m). Then, we can calculate $\vec{\rho}$ and ρ^m. We can treat $\widetilde{S}_+(m; r)$ in terms of operating $\widetilde{S}_+(m; \vec{\sigma}^0)$, which is defined in the following, on $|\min(n)\rangle$:

$$
\widetilde{S}_+(m; \vec{\sigma}^0) = \prod_{r=1}^{m} \left(\widetilde{S}_+(m; r) \right)^{\sigma^0(r)}, \tag{4.5}
$$

i.e.,

$$
|N; \vec{\rho}, \vec{\sigma}^0\rangle = \widetilde{S}_+(m; \vec{\sigma}^0)|N; \vec{\rho}, \vec{\sigma}\rangle \qquad (n = 2m), \tag{4.6a}
$$

$$
|N; \vec{\rho}, \rho_m, \vec{\sigma}^0\rangle = \widetilde{S}_+(m; \vec{\sigma}^0)|N; \vec{\rho}, \rho^m, \vec{\sigma}\rangle \qquad (n = 2m + 1), \tag{4.6b}
$$

$$
\vec{\sigma}^0 = \sigma^0(1), \sigma^0(2), \ldots, \sigma^0(m). \tag{4.7}
$$

Since $[\widetilde{S}_+(m; r), \widetilde{S}_+(m; r')] = 0$, the ordering of $\widetilde{S}_+(m; r)$ in $\widetilde{S}_+(m; \vec{\sigma}^0)$ is arbitrary. In the notation familiar to the $SU(2)$ algebra, $\sigma^0(r)$ is expressed as

$$
\sigma^0(r) = \sigma^r + \sigma_0^r. \tag{4.8}
$$

Of course, with the use of the angular momentum coupling rule, we are able to obtain the eigenstate of $(\widetilde{S}_{\pm, 0}(m))$.

Next, we consider some problems related to the quantum number ρ^r. We notice that $\widetilde{R}^{0,0}(m; r, k)$ defined in the relations (2.9) and (2.13) and their Hermitian conjugates $\widetilde{R}_{0,0}(m; r, k)$ and $\widetilde{R}_0(m; r)$ introduced in the relation (3.17a) form the $SU(m)$ algebras for the cases with $n = 2m$ and $n = 2m+1$. The detail is given in the appendix. First, we show that the state (4.6) is the minimum weight state of the $SU(m)$ algebra. For this, the following relations for the cases with $n = 2m$ and $2m + 1$ are useful:

$$
[\widetilde{R}_{0,0}(m; r, k), \widetilde{S}_+(m; r')] = (\delta_{rr'} - \delta_{kr'})\widetilde{R}_{1,-1}(m; r, k), \tag{4.9a}
$$

$$[\,\widetilde{R}_{1,-1}(m;r,k),\ \widetilde{S}_+(m;r')\,]=0, \tag{4.9b}$$

$$\widetilde{R}_{1,-1}(m;r,k)|\min(n)\rangle=0, \qquad \widetilde{R}_{0,0}(m;r,k)|\min(n)\rangle=0. \tag{4.9c}$$

The relation (4.9) leads us to

$$\widetilde{R}_{0,0}(m;r,k)|N;\vec{\rho},\vec{\sigma}^0\rangle=0, \tag{4.10a}$$

$$\widetilde{R}_{0,0}(m;r,k)|N;\vec{\rho},\rho^m,\vec{\sigma}^0\rangle=0. \tag{4.10b}$$

Further, we have

$$[\,\widetilde{R}_0(m;r),\ \widetilde{S}_+(m;r')\,]=0, \qquad \widetilde{R}_0(m;r)|\min(n)\rangle=-\rho^r|\min(n)\rangle. \tag{4.11}$$

Then, we obtain

$$\widetilde{R}_0(m;r)|N;\vec{\rho},\vec{\sigma}^0\rangle=-\rho^r|N;\vec{\rho},\vec{\sigma}^0\rangle, \tag{4.12a}$$

$$\widetilde{R}_0(m;r)|N;\vec{\rho},\rho^m,\vec{\sigma}^0\rangle=-\rho^r|N;\vec{\rho},\rho^m,\vec{\sigma}^0\rangle. \tag{4.12b}$$

The relations (4.10) and (4.12) tell us that the state (4.6) is the minimum weight state of the $SU(m)$ algebra.

Since we get the minimum weight states of the present $SU(m)$ algebra, we will consider $\widetilde{R}^{0,0}(m;r,k)$ operated on these states. First, we define the following raising operator:

$$\widetilde{R}^{0,0}(m;\vec{\lambda}^0)=\prod_{r=1}^{m-1}\left[\prod_{k=r+1}^{m}\left(\widetilde{R}^{0,0}(m;r,k)\right)^{\lambda^0(r,k)}\right], \tag{4.13}$$

$$\vec{\lambda}^0=\lambda^0(1,2),\lambda^0(1,3),\ldots,\lambda^0(1,m),\lambda^0(2,3),\lambda^0(2,4),\ldots,\lambda^0(2,m),\ldots,\lambda^0(m-1,m). \tag{4.14}$$

It should be noted that, since the operators $\widetilde{R}^{0,0}(m;r,k)$ are not mutually commuted, the ordering of their product is fixed beforehand. The total number of $\vec{\lambda}^0$ is equal to $m(m-1)/2$. Operating $\widetilde{R}^{0,0}(m;\vec{\lambda}^0)$ on the minimum weight states (4.6), we define the states

$$|N;\vec{\lambda}^0,\vec{\rho},\vec{\sigma}^0\rangle=\widetilde{R}^{0,0}(m;\vec{\lambda}^0)|N;\vec{\rho},\vec{\sigma}^0\rangle, \tag{4.15a}$$

$$|N;\vec{\lambda}^0,\vec{\rho},\rho^m,\vec{\sigma}^0\rangle=\widetilde{R}^{0,0}(m;\vec{\lambda}^0)|N;\vec{\rho},\rho^m,\vec{\sigma}^0\rangle. \tag{4.15b}$$

The states (4.15) are the eigenstates of $\widetilde{R}_0(m;r)$ given in the relation (4.2). After lengthy calculation, we can derive the commutation relation

$$[\,\widetilde{R}_0(m;r),\ \widetilde{R}^{0,0}(m;r',k')\,]=\begin{cases}\left(F_{r,r'}(m)-F_{r,k'}(m)\right)\widetilde{R}^{0,0}(m;r',k') & (4.16a)\\ \qquad\qquad (r=1,2,\ldots,m-1), & \\ 0 \qquad (r=m). & (4.16b)\end{cases}$$

The relation (4.16) is only applicable to the case with $n=2m+1$. With the use of the relation (4.16), we can show that the states (4.15) are the eigenstates of $\widetilde{R}_0(m;r)$, the eigenvalues of which we denote ρ_0^r $(r=1,2,\ldots,m-1,m)$:

$$\rho_0^r=\begin{cases}\displaystyle\sum_{r'=1}^{m-1}\left(\sum_{k'=r'+1}^{m}\left(F_{r,r'}(m)-F_{r,k'}(m)\right)\lambda(m';k')\right)-\rho^r & (4.17a)\\ \qquad\qquad (r=1,2,\ldots,m-1), & \\ -\rho^m \quad (r=m). & (4.17b)\end{cases}$$

In a form similar to that given in the relation (4.4), we will use the notation $\vec{\rho}_0$:

$$\vec{\rho}_0 = \rho_0^1, \rho_0^2, \ldots, \rho_0^{m-1}. \tag{4.18}$$

Let us pick up two states specified by $\vec{\lambda}^0(1)$ and $\vec{\lambda}^0(2)$ with $\vec{\lambda}^0(1) \neq \vec{\lambda}^0(2)$. If these two give us $\vec{\rho}^{\,0}(1)$ and $\vec{\rho}^{\,0}(2)$, respectively, with $\vec{\rho}^{\,0}(1) \neq \vec{\rho}^{\,0}(2)$, these two states are orthogonal to each other. But, if $\vec{\rho}^{\,0}(1) = \vec{\rho}^{\,0}(2)$, they are not always orthogonal. However, they are linearly independent of each other, because the states (4.15) are specified by $\vec{\lambda}^0$, the number of which is $m(m-1)/2$. It is equal to the total of the quantum numbers characterizing the $SU(m)$ algebra except the minimum weight states. In such cases, we must adopt some idea, for example the Schmidt method, for constructing the orthogonal set.

We must, further, investigate the role of the vectors and the spinors for constructing the linearly independent basis. These are defined in the relations (2.10)–(2.12) and (2.14)–(2.17). A simple idea for this task is to construct the raising operators by appropriate products of the vectors and the spinors, which are similar to the forms (4.5) and (4.13). With the help of this procedure, we can construct the linearly independent basis, which may be justified in the relations (2.23) and (2.24). However, for this task, we have a problem to be reconsidered. As was already mentioned, our present system contains two subalgebras, i.e., the $SU(2)$ and $SU(m)$ algebras. In our form, the vectors and the spinors are introduced as the tensor operators in the $SU(2)$ algebra. Up to now, we have not considered the relation between the tensors and the $SU(m)$ algebra. Therefore, not only from the side of the $SU(2)$ algebra, but also from the side of the $SU(m)$ algebra, it should be reconsidered. But, this consideration may be impossible to perform for the general case with arbitrary m. In the next section, some concrete results for the cases with $n = 2, 3, 4$, and 5 will be presented. Then, we will return to this point.

5. Some simple examples and discussion

In Sect. 6 of (I), we showed the minimum weight states of the cases with $n = 2, 3, 4$, and 5. In this section, mainly for the above cases, we will show the results obtained in this paper. First, we will treat the $SU(2)$ Lipkin model, i.e., $n = 2$ and $m = 1$. In this case, only one case with $r = 1$ appears and the index k is unnecessary. Three generators are given by

$$\widetilde{S}_+(1;1) = \widetilde{S}^1 \,(= \widetilde{S}_+), \quad \widetilde{S}_-(1;1) = \widetilde{S}_1 \,(= \widetilde{S}_-), \quad \widetilde{S}_0(1;1) = \frac{1}{2}\widetilde{S}_1^1 \,(= \widetilde{S}_0). \tag{5.1}$$

The orthogonal set $\{|ss_0\rangle\}$ is given in the relation

$$|ss_0\rangle = \sqrt{\frac{(s-s_0)!}{(2s)!(s+s_0)!}}\,\left(\widetilde{S}_+\right)^{s+s_0}|s\rangle. \tag{5.2}$$

Hereafter, we will omit the total fermion number N and the notations used in Sect. 6 of (I) will be adopted, for example, $\widetilde{S}_{\pm,0}$ in the relation (5.1).

Next, we will discuss the $SU(3)$ Lipkin model, i.e., $n = 3$ and $m = 1$. In this case, the case with $r = 1$ also appears and k is unnecessary. The eight generators are shown in the form

$$\widetilde{S}_+(1;1) = \widetilde{S}_1^2 \,(= \widetilde{S}_+), \quad \widetilde{S}_-(1;1) = \widetilde{S}_2^1 \,(= \widetilde{S}_-), \quad \widetilde{S}_0(1;1) = \frac{1}{2}\left(\widetilde{S}_2^2 - \widetilde{S}_1^1\right)\,(= \widetilde{S}_0), \tag{5.3a}$$

$$\widetilde{R}^{\frac{1}{2},+\frac{1}{2}}(1;1) = \widetilde{S}^2 \,(= \widetilde{R}^{\frac{1}{2},+\frac{1}{2}}), \quad \widetilde{R}^{\frac{1}{2},-\frac{1}{2}}(1;1) = \widetilde{S}^1 \,(= \widetilde{R}^{\frac{1}{2},-\frac{1}{2}}),$$

$$\widetilde{R}_{\frac{1}{2},+\frac{1}{2}}(1;1) = \widetilde{S}_2 \,(= \widetilde{R}_{\frac{1}{2},+\frac{1}{2}}), \quad \widetilde{R}_{\frac{1}{2},-\frac{1}{2}}(1;1) = \widetilde{S}_1 \,(= \widetilde{R}_{\frac{1}{2},-\frac{1}{2}}), \tag{5.3b}$$

$$\widetilde{\mathcal{P}}_0(1;1) = \frac{1}{2}\left(\widetilde{S}_2^2 + \widetilde{S}_1^1\right) \ (= \widetilde{R}_0). \tag{5.3c}$$

The minimum weight state $|\rho, \sigma\rangle$ obeys

$$\widetilde{S}_-|\rho,\sigma\rangle = \widetilde{R}_{\frac{1}{2},\pm\frac{1}{2}}|\rho,\sigma\rangle = 0, \quad \widetilde{S}_0|\rho,\sigma\rangle = -\sigma|\rho,\sigma\rangle, \quad \widetilde{R}_0|\rho,\sigma\rangle = -\rho|\rho,\sigma\rangle. \tag{5.4}$$

Therefore, we have the state with the eigenvalue (σ, σ_0) in the form

$$|\rho, \sigma\sigma_0\rangle = \sqrt{\frac{(\sigma - \sigma_0)!}{(2\sigma)!(\sigma + \sigma_0)!}} \left(\widetilde{S}_+\right)^{\sigma+\sigma_0} |\rho, \sigma\rangle. \tag{5.5}$$

With the use of the state (5.5), we define the following state:

$$|j, ss_0; \rho\sigma\rangle = \sum_{j_0\sigma_0} \langle jj_0\sigma\sigma_0|ss_0\rangle \widetilde{R}^{j,j_0}|\rho, \sigma\sigma_0\rangle, \tag{5.6}$$

$$\widetilde{R}^{j,j_0} = \sqrt{\frac{(2j)!}{(j+j_0)!(j-j_0)!}} \left(\widetilde{R}^{\frac{1}{2},+\frac{1}{2}}\right)^{j+j_0} \left(\widetilde{R}^{\frac{1}{2},-\frac{1}{2}}\right)^{j-j_0}. \tag{5.7}$$

Here, $\widetilde{R}^{j,j_0}$ denotes the tensor operator with rank j, i.e., $j = 1/2, 1, 3/2, \ldots$ and $j_0 = -j, -j + 1, \ldots, j - 1, j$. It is easily verified that $|j, ss_0; \rho\sigma\rangle$ satisfies

$$\widetilde{R}_0|j, ss_0; \rho\sigma\rangle = (3j - \rho)|j, ss_0; \rho\sigma\rangle, \tag{5.8a}$$

$$\widetilde{\boldsymbol{S}}^2|j, ss_0; \rho\sigma\rangle = s(s+1)|j, ss_0; \rho\sigma\rangle, \tag{5.8b}$$

$$\widetilde{S}_0|j, ss_0; \rho\sigma\rangle = s_0|j, ss_0; \rho\sigma\rangle, \tag{5.8c}$$

$$\widetilde{\Gamma}_{su(3)}|j, ss_0; \rho\sigma\rangle = \left(\sigma(\sigma+1) + \frac{1}{3}\rho(\rho+3)\right)|j, ss_0; \rho\sigma\rangle. \tag{5.8d}$$

Therefore, we can learn that the set $\{|j, ss_0; \rho\sigma\rangle\}$ is orthogonal and, as is well known, this set is specified by five quantum numbers characterizing the $SU(3)$ algebra.

Our third concern is related to the $SU(4)$ Lipkin model, i.e., $n = 4$ and $m = 2$. For $r = 1$ and 2, the $SU(2)$ subalgebras can be expressed as

$$\widetilde{S}_+(2;1) = \widetilde{S}_2^3 \ (= \widetilde{S}_+(1)), \quad \widetilde{S}_-(2;1) = \widetilde{S}_3^2 \ (= \widetilde{S}_-(1)), \quad \widetilde{S}_0(2;1) = \frac{1}{2}\left(\widetilde{S}_3^3 - \widetilde{S}_2^2\right) \ (= \widetilde{S}_0(1)), \tag{5.9a}$$

$$\widetilde{S}_+(2;2) = \widetilde{S}^1 \ (= \widetilde{S}_+(2)), \quad \widetilde{S}_-(2;2) = \widetilde{S}_1 \ (= \widetilde{S}_-(2)), \quad \widetilde{S}_0(2;2) = \frac{1}{2}\widetilde{S}_1^1 \ (= \widetilde{S}_0(2)). \tag{5.9b}$$

The addition can be denoted as

$$\widetilde{S}_{\pm,0} = \widetilde{S}_{\pm,0}(1) + \widetilde{S}_{\pm,0}(2). \tag{5.10}$$

The scalar and the vector operators, which appear in the case with ($r = 1$, $k = 2$), are written down as

$$\widetilde{R}^{0,0}(2;1,2) = \widetilde{S}_1^3 + \widetilde{S}^2 \ (= \widetilde{R}_+), \quad \widetilde{R}_{0,0}(2;1,2) = \widetilde{S}_3^1 + \widetilde{S}_2 \ (= \widetilde{R}_-), \tag{5.11}$$

$$\widetilde{R}^{1,+1}(2;1,2) = -\widetilde{S}^3 \ (= \widetilde{R}^{1,+1}), \quad \widetilde{R}_{1,+1}(2;1,2) = -\widetilde{S}_3 \ (= \widetilde{R}_{1,+1}), \tag{5.12}$$

$$\widetilde{R}^{1,0}(2;1,2) = \frac{1}{\sqrt{2}}\left(\widetilde{S}_1^3 - \widetilde{S}^2\right) \ (= \widetilde{R}^{1,0}), \quad \widetilde{R}_{1,0}(2;1,2) = \frac{1}{\sqrt{2}}\left(\widetilde{S}_3^1 - \widetilde{S}_2\right) \ (= \widetilde{R}_{1,0}), \tag{5.13}$$

$$\widetilde{R}^{1,-1}(2;1,2) = \widetilde{S}_1^2 \ (= \widetilde{R}^{1,-1}), \quad \widetilde{R}_{1,-1}(2;1,2) = \widetilde{S}_2^1 \ (= \widetilde{R}_{1,-1}). \tag{5.14}$$

The operator $\widetilde{\mathcal{P}}_0(2;1)$ can be expressed as

$$\widetilde{\mathcal{P}}_0(2;1) = \frac{1}{2} \left(\widetilde{S}_3^3 + \widetilde{S}_2^2 - \widetilde{S}_1^1 \right) \ (= \widetilde{R}_0). \tag{5.15}$$

The above are 15 generators of the $SU(4)$ algebra. In Sect. 4, we mentioned that the scalar operators form the $SU(m)$ algebra, in which the present case is given by the $SU(2)$ algebra $(\widetilde{R}_{\pm,0})$. Of course, we have

$$\left[\text{ any of } \widetilde{R}_{\pm,0}, \text{ any of } \widetilde{S}_{\pm,0} \right] = 0. \tag{5.16}$$

Then, as the Hermitian operators commuting with the Casimir operator $\widetilde{\Gamma}_{SU(4)}$, we can choose $(\widetilde{\boldsymbol{S}}^2, \widetilde{S}_0)$ and $(\widetilde{\boldsymbol{R}}^2, \widetilde{R}_0)$. By putting $\sigma_0 = -\sigma$ in the state (6.13) of (I), we obtain the state $|\rho, \sigma^1, \sigma^2, \sigma\sigma_0 = -\sigma\rangle$ and, then, the eigenstate for $\widetilde{\boldsymbol{R}}^2, \widetilde{R}_0, \widetilde{\boldsymbol{S}}^2$, and $\widetilde{S}_0$:

$$|\rho\rho_0, \sigma^1, \sigma^2, \sigma\sigma_0\rangle = \sqrt{\frac{(\rho - \rho_0)!}{(2\rho)!(\rho + \rho_0)!}} \sqrt{\frac{(\sigma - \sigma_0)!}{(2\sigma)!(\sigma + \sigma_0)!}}$$
$$\times \left(\widetilde{R}_+ \right)^{\rho+\rho_0} \left(\widetilde{S}_+ \right)^{\sigma+\sigma_0} |\rho, \sigma^1, \sigma^2, \sigma\sigma_0 = -\sigma\rangle. \tag{5.17}$$

The present case is the simplest: $m = 1$. Therefore, the state (5.17) is easily obtained.

Now, we must discuss the vector operators (5.12)–(5.14). As was already mentioned, there exist two $SU(2)$ algebras satisfying the relation (5.16). We started the present form in the $SU(2)$ algebra $(\widetilde{S}_{\pm,0})$. However, at the present stage, we have no reason to give the treatment for $(\widetilde{S}_{\pm,0})$ priority over that for $(\widetilde{R}_{\pm,0})$. In other words, we must treat these two on an equal footing. Although the vector operators (5.12)–(5.14) are given for $(\widetilde{S}_{\pm,0})$, they satisfy the relation

$$\left[\widetilde{R}_+, \widetilde{R}^{1,\nu} \right] = 0, \quad \left[\widetilde{R}_0, \widetilde{R}^{1,\nu} \right] = \nu \widetilde{R}^{1,\nu} \quad (\nu = \pm 1, \ 0). \tag{5.18}$$

Then, by calculating $[\widetilde{R}_-, \widetilde{R}^{1,\nu}]$ and $[\widetilde{R}_-, [\widetilde{R}_-, \widetilde{R}^{1,\nu}]]$, we can present the vector operators for $(\widetilde{R}_{\pm,0})$:

$$\widetilde{R}^{1,1;1,1} = -\widetilde{S}^3 \ (= \widetilde{R}^{1,1}), \quad \widetilde{R}^{1,1;1,0} = \frac{1}{\sqrt{2}} \left(\widetilde{S}_2^3 - \widetilde{S}_1^1 \right) \ \left(= \frac{1}{\sqrt{2}} \left(\widetilde{S}_+(1) - \widetilde{S}_+(2) \right) \right),$$

$$\widetilde{R}^{1,1;1,-1} = \widetilde{S}_2^1 \ (= \widetilde{R}_{1,-1}), \tag{5.19}$$

$$\widetilde{R}^{1,0;1,1} = \frac{1}{\sqrt{2}} \left(\widetilde{S}_1^3 - \widetilde{S}^2 \right) \ (= \widetilde{R}^{1,0}),$$

$$\widetilde{R}^{1,0;1,0} = -\frac{1}{\sqrt{2}} \left(\widetilde{S}_3^3 - \widetilde{S}_2^2 - \widetilde{S}_1^1 \right) \ \left(= - \left(\widetilde{S}_0(1) - \widetilde{S}_0(2) \right) \right),$$

$$\widetilde{R}^{1,0;1,-1} = \frac{1}{\sqrt{2}} \left(\widetilde{S}_3^1 - \widetilde{S}^2 \right) \ (= \widetilde{R}_{1,0}), \tag{5.20}$$

$$\widetilde{R}^{1,-1;1,1} = \widetilde{S}_1^2 \ (= \widetilde{R}^{1,-1}), \quad \widetilde{R}^{1,-1;1,0} = -\frac{1}{\sqrt{2}} \left(\widetilde{S}_3^2 - \widetilde{S}_1 \right) \ \left(= -\frac{1}{\sqrt{2}} \left(\widetilde{S}_-(1) - \widetilde{S}_-(2) \right) \right),$$

$$\widetilde{R}^{1,-1;1,-1} = -\widetilde{S}_3 \ (= \widetilde{R}_{1,1}). \tag{5.21}$$

Here, $\widetilde{R}^{1,\mu;1,\nu}$ $(\mu, \nu = \pm 1, 0)$ denotes the tensor operator with $(1, \mu)$ for $(\widetilde{S}_{\pm,0})$ and $(1, \nu)$ for $(\widetilde{R}_{\pm,0})$. The operator $\widetilde{R}^{1,\mu;1,\nu}$ satisfies

$$\left(\widetilde{R}^{1,\mu;1,\nu} \right)^* = (-)^{\mu+\nu} \widetilde{R}_{1,-\mu;1,-\nu}. \tag{5.22}$$

Therefore, we have

$$\sum_{\mu\nu}(-)^{1-\mu}(-)^{1-\nu}\widetilde{R}^{1,\mu;1,\nu}\widetilde{R}^{1,-\mu;1,-\nu} = \sum_{\mu\nu}\widetilde{R}^{1,\mu;1,\nu}\widetilde{R}_{1,\mu;1,\nu} = \sum_{\mu\nu}\widetilde{R}_{1,\mu;1,\nu}\widetilde{R}^{1,\mu;1,\nu}$$

$$= 2\widetilde{\Gamma}_{SU(4)} - \widetilde{\boldsymbol{S}}^2 - \widetilde{\boldsymbol{R}}^2. \tag{5.23}$$

Here, we used the relations (5.19)–(5.21).

In order to obtain the linearly independent basis for the orthogonal set for the $SU(4)$ algebra, the basis should be expressed in terms of nine parameters (if possible, quantum numbers) including $(\rho, \sigma^1, \sigma^2)$. In the state $|\rho\rho_0, \sigma^1, \sigma^2, \sigma\sigma_0\rangle$ given in the relation (5.17), six quantum numbers appear. Therefore, the linearly independent basis may be obtained by applying a certain operator expressed in terms of three parameters on the state (5.17). To search for this operator, it may be important to treat $(\widetilde{R}_{\pm,0})$ on an equal footing with $(\widetilde{S}_{\pm,0})$. First, we note the relation

$$\left[\,\widetilde{S}_+,\,\left(-\widetilde{S}^3\right)^l\,\right] = \left[\,\widetilde{R}_+,\,\left(-\widetilde{S}^3\right)^l\,\right] = 0, \tag{5.24a}$$

$$\left[\,\widetilde{S}_0,\,\left(-\widetilde{S}^3\right)^l\,\right] = \left[\,\widetilde{R}_0,\,\left(-\widetilde{S}^3\right)^l\,\right] = l\left(-\widetilde{S}^3\right)^l \quad (l = 0, 1, 2, \ldots). \tag{5.24b}$$

The relation (5.24) suggests that we can construct the tensor operator with rank l which is common to $(\widetilde{S}_{\pm,0})$ and $(\widetilde{R}_{\pm,0})$, and if we denote it as $\widetilde{R}^{l,l_0;l,\lambda_0}$, it can be given in the form

$$\widetilde{R}^{l,l_0;l,\lambda_0} = \sqrt{\frac{(l+l_0)!(l-l_0)!}{(2l)!}}\sqrt{\frac{(l+\lambda_0)!(l-\lambda_0)!}{(2l)!}}\left(\vec{R}_-\right)^{l-\lambda_0}\left(\vec{S}_-\right)^{l-l_0}\left(-\widetilde{S}^3\right)^l. \tag{5.25}$$

Here, we used the notations for $\widetilde{O}$ and $\widetilde{A}$ in the form

$$\left(\vec{O}\right)^n\widetilde{A} = \underbrace{[\widetilde{O},\ldots,[\widetilde{O},[\widetilde{O},\widetilde{A}]]\ldots]}_{n}. \tag{5.26}$$

Then, operating $\widetilde{R}^{l,l_0;l,\lambda_0}$ on the state $|\rho\rho_0, \sigma^1, \sigma^2, \sigma\sigma_0\rangle$, we have

$$|l\lambda_0, ll_0, \rho\rho_0, \sigma^1, \sigma^2, \sigma\sigma_0\rangle = \widetilde{R}^{l,l_0;l,\lambda_0}|\rho\rho_0, \sigma^1, \sigma^2, \sigma\sigma_0\rangle. \tag{5.27}$$

Certainly, $\widetilde{R}^{l,l_0;l,\lambda_0}$ is expressed in terms of three parameters. With the use of the Clebsch–Gordan coefficients, we obtain the eigenstate of $\widetilde{\boldsymbol{S}}^2$, $\widetilde{S}_0$, $\widetilde{\boldsymbol{R}}^2$, $\widetilde{R}_0$, and $\widetilde{\Gamma}_{su(4)}$:

$$|l, rr_0, ss_0, \sigma; \rho\sigma^1\sigma^2\rangle = \sum_{\lambda_0\rho_0}\sum_{l_0\sigma_0}\langle l\lambda_0\rho\rho_0|rr_0\rangle\langle ll_0\sigma\sigma_0|ss_0\rangle|l\lambda_0, ll_0, \rho\rho_0, \sigma^1, \sigma^2, \sigma\sigma_0\rangle,$$

$$l = |r-\rho|,\ldots,(r+\rho) \quad \text{and} \quad |s-\sigma|,\ldots,(s+\sigma). \tag{5.28}$$

The eigenvalues are given by $s(s+1)$, s_0, $r(r+1)$, r_0, and $\sigma^1(\sigma^1+1)+\sigma^2(\sigma^2+1)+\rho(\rho+4)/2$, respectively. Except for l, the eight parameters are the quantum numbers. Of course, $\sigma = |\sigma^1 - \sigma^2|,\ldots,(\sigma^1+\sigma^2)$. Therefore, the set composed of the states (5.28) may be the linearly independent basis for the $SU(4)$ Lipkin model.

Finally, we will discuss the case with $n = 5$, i.e., $m = 2$. This case also contains two $SU(2)$ algebras:

$$\widetilde{S}_+(2;1) = \widetilde{S}_3^4 \,(=\widetilde{S}_+(1)), \quad \widetilde{S}_-(2;1) = \widetilde{S}_4^3 \,(=\widetilde{S}_-(1)), \quad \widetilde{S}_0(2;1) = \frac{1}{2}\left(\widetilde{S}_4^4 - \widetilde{S}_3^3\right) \,(=\widetilde{S}_0(1)),$$
$$\tag{5.29a}$$

$$\widetilde{S}_+(2;2) = \widetilde{S}_1^2 \ (= \widetilde{S}_+(2)), \quad \widetilde{S}_-(2;2) = \widetilde{S}_2^1 \ (= \widetilde{S}_-(2)), \quad \widetilde{S}_0(2;2) = \frac{1}{2}\left(\widetilde{S}_2^2 - \widetilde{S}_1^1\right) \ (= \widetilde{S}_0(2)).$$
(5.29b)

Then, the addition of the two is expressed as

$$\widetilde{S}_{\pm,0} = \widetilde{S}_{\pm,0}(1) + \widetilde{S}_{\pm,0}(2).$$
(5.30)

The scalar operators are as follows:

$$\widetilde{R}^{0,0}(2;1,2) = \widetilde{S}_2^4 + \widetilde{S}_1^3 \ (= \widetilde{R}_+), \quad \widetilde{R}_{0,0}(2;1,2) = \widetilde{S}_4^2 + \widetilde{S}_3^1 \ (= \widetilde{R}_-).$$
(5.31)

In the present case, two spinors appear:

$$\widetilde{R}^{\frac{1}{2},+\frac{1}{2}}(2;1) = \widetilde{S}^4 \ (= \widetilde{R}^{\frac{1}{2},+\frac{1}{2}}(1)), \quad \widetilde{R}^{\frac{1}{2},-\frac{1}{2}}(2;1) = \widetilde{S}^3 \ (= \widetilde{R}^{\frac{1}{2},-\frac{1}{2}}(1)),$$

$$\widetilde{R}_{\frac{1}{2},+\frac{1}{2}}(2;1) = \widetilde{S}_4 \ (= \widetilde{R}_{\frac{1}{2},+\frac{1}{2}}(1)), \quad \widetilde{R}_{\frac{1}{2},-\frac{1}{2}}(2;1) = \widetilde{S}_3 \ (= \widetilde{R}_{\frac{1}{2},-\frac{1}{2}}(1)),$$
(5.32a)

$$\widetilde{R}^{\frac{1}{2},+\frac{1}{2}}(2;2) = \widetilde{S}^2 \ (= \widetilde{R}^{\frac{1}{2},+\frac{1}{2}}(2)), \quad \widetilde{R}^{\frac{1}{2},-\frac{1}{2}}(2;2) = \widetilde{S}^1 \ (= \widetilde{R}^{\frac{1}{2},-\frac{1}{2}}(2)),$$

$$\widetilde{R}_{\frac{1}{2},+\frac{1}{2}}(2;2) = \widetilde{S}_2 \ (= \widetilde{R}_{\frac{1}{2},+\frac{1}{2}}(2)), \quad \widetilde{R}_{\frac{1}{2},-\frac{1}{2}}(2;2) = \widetilde{S}_1 \ (= \widetilde{R}_{\frac{1}{2},-\frac{1}{2}}(2)).$$
(5.32b)

The vector operators are given as

$$\widetilde{R}^{1,+1}(2;1,2) = -\widetilde{S}_1^4 \ (= \widetilde{R}^{1,+1}), \quad \widetilde{R}_{1,+1}(2;1,2) = -\widetilde{S}_4^1 \ (= \widetilde{R}_{1,+1}),$$
(5.33)

$$\widetilde{R}^{1,+0}(2;1,2) = \frac{1}{\sqrt{2}}\left(\widetilde{S}_2^4 - \widetilde{S}_1^3\right), \quad \widetilde{R}_{1,0}(2;1,2) = \frac{1}{\sqrt{2}}\left(\widetilde{S}_4^2 - \widetilde{S}_3^1\right),$$
(5.34)

$$\widetilde{R}^{1,-1}(2;1,2) = \widetilde{S}_2^3 \ (= \widetilde{R}^{1,-1}), \quad \widetilde{R}_{1,-1}(2;1,2) = \widetilde{S}_3^2 \ (= \widetilde{R}_{1,-1}).$$
(5.35)

The operators $\widetilde{\mathcal{P}}_0(2;1)$ and $\widetilde{\mathcal{P}}_0(2;2)$ can be expressed as

$$\widetilde{\mathcal{P}}_0(2;1) = \frac{1}{2}\left(\widetilde{S}_4^4 + \widetilde{S}_3^3\right), \qquad \widetilde{\mathcal{P}}_0(2;2) = \frac{1}{2}\left(\widetilde{S}_2^2 + \widetilde{S}_1^1\right),$$
(5.36)

i.e.,

$$\widetilde{R}_0(1) = \widetilde{\mathcal{P}}_0(2;1) - \widetilde{\mathcal{P}}_0(2;2) = \frac{1}{2}\left(\widetilde{S}_4^4 + \widetilde{S}_3^3 - \widetilde{S}_2^2 - \widetilde{S}_1^1\right) \ (= \widetilde{R}_0),$$
(5.37a)

$$\widetilde{R}_0(2) = \widetilde{\mathcal{P}}_0(2;1) + \widetilde{\mathcal{P}}_0(2;2) = \frac{1}{2}\left(\widetilde{S}_4^4 + \widetilde{S}_3^3 + \widetilde{S}_2^2 + \widetilde{S}_1^1\right).$$
(5.37b)

With the use of the above relations, we will show the results for the case with $n = 5$ simply in comparative argument with the case with $n = 4$.

In (I), the minimum weight state in the present case is given by $|\rho^1, \rho^2, \sigma^1, \sigma^2\rangle$, where ρ^1, ρ^2, σ^1, and σ^2 denote the eigenvalues of $\widetilde{R}_0(1) \ (= \widetilde{R}_0)$, $\widetilde{R}_2(2)$, $\widetilde{S}_0(1)$, and $\widetilde{S}_0(2)$, respectively. The set $(\widetilde{R}_{\pm,0})$ also forms the $SU(2)$ algebra obeying the same relation as that given in the relation (5.16), and there also exists one set of the vector for $(\widetilde{S}_{\pm,0})$. Therefore, the treatment for these two may be similar to the case with $n = 4$. The difference between these two can be found in the appearance of the Hermitian operator $\widetilde{R}_0(2)$ and two sets of spinors. Under the same idea as in the case with $n = 4$, we introduce $\widetilde{R}^{\frac{1}{2},\mu;\frac{1}{2},\nu}$:

$$\widetilde{R}^{\frac{1}{2},+\frac{1}{2};\frac{1}{2},+\frac{1}{2}} = \widetilde{S}^4, \quad \widetilde{R}^{\frac{1}{2},-\frac{1}{2};\frac{1}{2},+\frac{1}{2}} = \widetilde{S}^3, \quad \widetilde{R}^{\frac{1}{2},+\frac{1}{2};\frac{1}{2},-\frac{1}{2}} = \widetilde{S}^2, \quad \widetilde{R}^{\frac{1}{2},-\frac{1}{2};\frac{1}{2},-\frac{1}{2}} = \widetilde{S}^1.$$
(5.38)

Then, we can define the following scalar operator for $(\widetilde{S}_{\pm,0})$ and $(\widetilde{R}_{\pm,0})$:

$$\widetilde{R}^{0,0;0,0} = \frac{1}{2}\sum_{\mu\nu}(-)^{\frac{1}{2}-\mu}(-)^{\frac{1}{2}-\nu}\widetilde{R}^{\frac{1}{2},\mu;\frac{1}{2},\nu}\widetilde{R}^{\frac{1}{2},-\mu;\frac{1}{2},-\nu} = \widetilde{S}^4\widetilde{S}^1 - \widetilde{S}^3\widetilde{S}^2. \tag{5.39}$$

The above should be compared with the relation (5.23). Further, we have the operator $\widetilde{R}^{j,j_0;j,\kappa_0}$:

$$\widetilde{R}^{j,j_0;j,\kappa_0} = \sqrt{\frac{(j+j_0)!(j-j_0)!}{(2j)!}} \cdot \sqrt{\frac{(j+\kappa_0)!(j-\kappa_0)!}{(2j)!}} \left(\overrightarrow{R}_-\right)^{j-\kappa_0}\left(\overrightarrow{S}_-\right)^{j-j_0}\left(\widetilde{S}^4\right)^{2j}$$

$$\left(j = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots\right). \tag{5.40}$$

The operators (5.39) and (5.40) satisfy

$$\left[\widetilde{R}_0(2),\ \left(\widetilde{R}^{0,0;0,0}\right)^{n_0}\widetilde{R}^{j,j_0;j,\kappa_0}\right] = 5(n_0+j)\left(\widetilde{R}^{0,0;0,0}\right)^{n_0}\widetilde{R}^{j,j_0;j,\kappa_0}. \tag{5.41}$$

In this connection, we have

$$\left[\widetilde{R}_0(2),\ \widetilde{R}^{l,l_0;l,\lambda_0}\right] = \left[\widetilde{R}_0(2),\ \widetilde{R}_+\right] = \left[\widetilde{R}_0(2)\ ,\ \widetilde{S}_+(1)\right] = \left[\widetilde{R}_0(2),\ \widetilde{S}_+(2)\right] = 0. \tag{5.42}$$

Of course, $\widetilde{R}^{l,l_0;l,\lambda_0}$ is obtained by replacing $(-\widetilde{S}^3)$ with $\widetilde{S}^4$ in the relation (5.25).

The above operators give us the following state:

$$|ll_0\lambda_0, n_0jj_0\kappa_0, \rho^1\rho_0^1, \rho^2, \sigma^1\sigma_0^1, \sigma^2\sigma_0^2\rangle$$
$$=\left(\widetilde{R}^{l,l_0;l,\lambda_0}\right)\left[\left(\widetilde{R}^{0,0;0,0}\right)^{n_0}\widetilde{R}^{j,j_0;j,\kappa_0}\right]\left(\widetilde{R}_+\right)^{\rho^1+\rho_0^1}\left(\widetilde{S}_+(1)\right)^{\sigma^1+\sigma_0^1}\left(\widetilde{S}_+(2)\right)^{\sigma^2+\sigma_0^2}|\rho^1,\rho^2,\sigma^1,\sigma^2\rangle. \tag{5.43}$$

We can see that the state (5.43) is expressed in terms of 14 parameters including ρ^1, ρ^2, σ^1, and σ^2. By using the angular momentum coupling rule, the state (5.43) can be rewritten to the form with the eigenvalues of $\widetilde{\boldsymbol{S}}^2$, $\widetilde{S}_0$, $\widetilde{\boldsymbol{R}}^2$, and $\widetilde{R}_0$, which is similar to the state (5.28). Of course, the states (5.43) form the linearly independent basis for the $SU(5)$ Lipkin model. The relations (5.41) and (5.42) teach us that the state (5.43) is the eigenstate of $\widetilde{R}_0(2)$ with the eigenvalue $5(n_0+j) - \rho^2$. The above is an outline of our idea for the $SU(5)$ Lipkin model.

Through the above arguments, we can learn some characteristic features of the Lipkin model. Classification of the model into two cases with $n = 2m$ and $2m+1$ may be essential for understanding the features demonstrated by the model. The number of $SU(2)$ subalgebras is equal to m for both cases and, further, they contain the $SU(m)$ subalgebra. In the case with $n = 2m+1$, there exists the degree of freedom related to $\widetilde{R}_0(m)$ and its eigenvalues are influenced by the spinors, which do not exist in the case with $n = 2m$.

6.　Concluding remarks

In these two papers, we have discussed formal aspects of the $SU(n)$ Lipkin model in rather general form. In (I), we were mainly concerned with the determination of the minimum weight states. In (II), after re-forming the model in the frame of spherical tensor representation, we re-formulated the model by classifying it into two cases: $n = 2m$ and $2m + 1$. In particular, in Sect. 5, we gave some concrete results for the cases with $n = 2, 3, 4,$ and 5, for which, in (I), we discussed only the
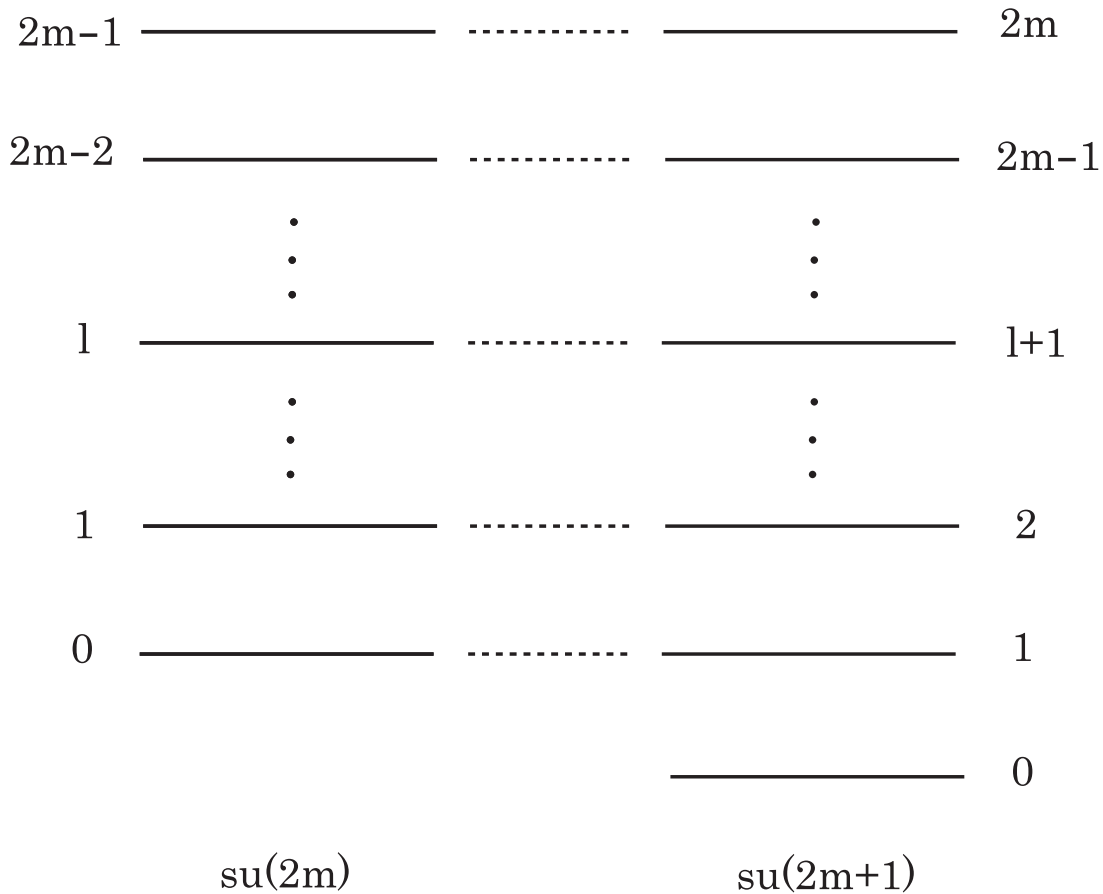
Fig. 1. The single-particle level schemes depicted schematically in the case of the $SU(2m)$ Lipkin model (left) and the $SU(2m + 1)$ Lipkin model (right).

minimum weight states. As a concluding remark, we will show the connection between the $SU(2m)$ and $SU(2m + 1)$ Lipkin models directly.

Concerning the single-particle level schemes, we regard these two as the models illustrated in Fig. 1. Let the single-particle level specified by the integer l ($l = 0, 1, 2, \ldots, 2m - 1$) in the $SU(2m)$ Lipkin model be shifted to the integer $(l + 1)$ in the $SU(2m + 1)$ Lipkin model. This situation resembles that of the level schemes of two supersymmetric partner potentials $V_1(x) = W^2(x) - W'(x)/\sqrt{2}$ and $V_2(x) = W^2(x) + W'(x)/\sqrt{2}$ with unit mass in one-dimensional supersymmetric quantum mechanics (see, for example, [4]). The investigation of the relation between the Lipkin models with $SU(2m)$ and $SU(2m + 1)$ algebraic structures and supersymmetric quantum mechanics may be one for interesting future studies if the relation exists.

The zeroth level cannot have any counterpart in the $SU(2m)$ Lipkin model. As a possible candidate, the following correspondence may be permitted:

$$\widetilde{S}^p \longrightarrow \widetilde{S}_1^{p+1}, \quad \widetilde{S}_p \longrightarrow \widetilde{S}_{p+1}^1, \quad \widetilde{S}_q^p \longrightarrow \widetilde{S}_{q+1}^{p+1} - \delta_{pq}\widetilde{S}_1^1 \quad (p, q = 1, 2, \ldots, 2m - 1). \quad (6.1)$$

The set { $\widetilde{S}_1^{p+1}$, $\widetilde{S}_{p+1}^1$, $\widetilde{S}_{q+1}^{p+1}$ } forms the $SU(2m)$ algebra and it becomes the $U(2m)$ algebra, if $\sum_{p=1}^{2m} \widetilde{S}_p^p$ is included. Further, if the operators $\widetilde{S}^p$ and $\widetilde{S}_p$ ($p = 1, 2, \ldots, 2m$) are included, it becomes the $SU(2m + 1)$ algebra. In fact, the total number of generators is given by

$$[(2m)^2 - 1] + 1 + 2 \cdot (2m) = (2m + 1)^2 - 1. \quad (6.2)$$

Under the correspondence (6.1), $\widetilde{\Gamma}_{SU(2m)}$ given in the relation (3.1) leads us to

$$\widetilde{\Gamma}_{SU(2m)} \longrightarrow \frac{1}{2}\left[\sum_{p=2}^{2m}\sum_{q=1}^{p-1}\left[\widetilde{S}_q^p,\,\widetilde{S}_p^q\right]_+ + \sum_{p=1}^{2m}\left(\widetilde{S}_p^p\right)^2 - \frac{1}{2m}\left(\sum_{p=1}^{2m}\widetilde{S}_p^p\right)^2\right]. \tag{6.3}$$

Adding the operators $\widetilde{S}^p$, $\widetilde{S}_p$, and $\sum_{p=1}^{2m}\widetilde{S}_p^p$ to the form (6.3) in terms of $\sum_{p=1}^{2m}[\widetilde{S}^p,\,\widetilde{S}_p]_+/2$ and $(1/(2m(2m+1)))(\sum_{p=1}^{2m}\widetilde{S}_p^p)^2/2$, we obtain $\widetilde{\Gamma}_{SU(2m+1)}$.

As the $SU(m)$ subalgebra, we treated the simplest case, $m=2$. In the near future, we will report some results in the cases with $m>2$. It may be interesting to investigate how the cases with $m=3$ and 4 relate to the Lipkin models with $n=3$ and 4, which were discussed in this paper.

Acknowledgements

Appendix. The $SU(m)$ algebra in the $SU(n)$ Lipkin model

The aim of this appendix is to show that, as a subalgebra, the $SU(n)$ Lipkin model contains the $SU(m)$ algebra. The discussion starts in the case with $n=2m$. We define the following set of operators specified by p and q $(p,q=1,2,\ldots,m-1)$:

$$\widetilde{R}^p(m) = \widetilde{S}_1^{2m-2p+1} + \widetilde{S}^{2m-2p}, \qquad \widetilde{R}_p(m) = \widetilde{S}_{2m-2p+1}^1 + \widetilde{S}_{2m-2p}, \tag{A.1a}$$

$$\widetilde{R}_q^p(m) = \widetilde{S}_{2m-2q+1}^{2m-2p+1} + \widetilde{S}_{2m-2q}^{2m-2p} - \delta_{pq}\widetilde{S}_1^1. \tag{A.1b}$$

The above set obeys the commutation relation

$$\left[\widetilde{R}^p(m),\,\widetilde{R}_q(m)\right] = \widetilde{R}_q^p(m), \tag{A.2a}$$

$$\left[\widetilde{R}_q^p(m),\,\widetilde{R}^r(m)\right] = \delta_{qr}\widetilde{R}^p(m) + \delta_{pq}\widetilde{R}^r(m), \tag{A.2b}$$

$$\left[\widetilde{R}_q^p(m),\,\widetilde{R}_r^s(m)\right] = \delta_{qs}\widetilde{R}_r^p(m) - \delta_{pr}\widetilde{R}_q^s(m). \tag{A.2c}$$

If $\widetilde{R}$ and m in the relation (A.2) read $\widetilde{S}$ and n in the relation (2.3) in (I), respectively, we can see that the set (A.1) obeys the $SU(m)$ algebra. Of course, the total number of generators is equal to $(m-1)^2 + 2(m-1) = m^2 - 1$.

In the case with $p < q$, let p and q in the relation (A.1a) read r and k, respectively, $(r=1,2,\ldots,m-1,\ k=r+1,r+2,\ldots,m-1)$. Then, $\widetilde{R}_q^p(m)$ becomes $\widetilde{R}^{0,0}(m;r,k)$ given in the relation (2.9) for the case with $k=r+1,r+2,\ldots,m-1$. Further, we can regard $\widetilde{R}^p(m)$ as $\widetilde{R}^{0,0}(m;r,m)$. In the case with $p > q$, if p and q are exchanged with each other, we have the Hermitian conjugate of $\widetilde{R}^{0,0}(m;r,k)$, i.e., $\widetilde{R}_{0,0}(m;r,k)$. In the case with $p=q$, p reads r, and then $\widetilde{R}_p^p(m)$ reduces to $2\widetilde{\mathcal{P}}_0(m;r)$ given in the relation (3.22). Of course, $\widetilde{\mathcal{P}}_0(m;r)$ is essentially equivalent to $\widetilde{R}_0(m;r)$. Thus, in the case with $n=2m$, we have the $SU(m)$ subalgebra in the $SU(n)$ Lipkin model.

Next, we discuss the case with $n=2m+1$. First, we define the following set of operators specified by p and q $(p,q=1,2,\ldots,m)$:

$$\widetilde{R}_q^p(m) = \widetilde{S}_{2m-2q+2}^{2m-2p+2} + \widetilde{S}_{2m-2q+1}^{2m-2p+1}. \tag{A.3}$$

We have the commutation relation

$$[\widetilde{R}_q^p(m), \ \widetilde{R}_r^s(m)] = \delta_{qs}\widetilde{R}_r^p(m) - \delta_{pr}\widetilde{R}_q^s(m). \tag{A.4}$$

The relation (A.4) tells us that the set $\{\widetilde{R}_q^p(m)\}$ obeys the well-known $U(m)$ algebra. In order to get the $SU(m)$ algebra, we define the following operator:

$$\overline{R}_q^p(m) = \widetilde{R}_q^p(m) - \delta_{pq}\frac{1}{m}\sum_{r=1}^{m}\widetilde{R}_r^r(m). \tag{A.5}$$

The set (A.5) satisfies

$$[\overline{R}_q^p(m), \ \overline{R}_r^s(m)] = \delta_{qs}\overline{R}_r^p(m) - \delta_{pr}\overline{R}_q^s(m), \tag{A.6a}$$

$$\sum_{p=1}^{m}\overline{R}_p^p(m) = 0. \tag{A.6b}$$

The relation (A.6b) tells us that, in the type $\overline{R}_p^p(m)$, we can choose $(m-1)$ independent operators. Therefore, the set $\{\overline{R}_q^p(m)\}$ obeys the $SU(m)$ algebra. Of course, the total number of generators is equal to $m^2 - 1$.

In the case with $p < q$, if p and q read r and k, respectively, $\overline{R}_q^p(m) (= \widetilde{R}_q^p(m))$ becomes $\widetilde{R}^{0,0}(m; r, k)$. In the case with $p > q$, if p and q are exchanged each other, we obtain the Hermitian conjugate of $\widetilde{R}^{0,0}(m; r, k)$, i.e., $\widetilde{R}_{0,0}(m; r, k)$. In the case with $p = q$, p reads r and then $\overline{R}_p^p(m)$ reduces to $2(\widetilde{P}_0(m; r) - (1/m) \cdot \sum_{r'=1}^{m}\widetilde{P}_0(m; r'))$. With the use of $C_{r,r'}(m)$ given in the relation (3.10), the relation (A.5) gives us

$$\sum_{r'=1}^{m}C_{r,r'}(m)\overline{R}_{r'}^{r'}(m) = 2\left(1 - \delta_{r,m}\right)\widetilde{Q}_0(m; r). \tag{A.7}$$

Therefore, we have

$$\widetilde{Q}_0(m; r) = \frac{1}{2}\sum_{r'=1}^{m}C_{r,r'}(m)\overline{R}_{r'}^{r'}(m) \qquad \text{for} \quad r = 1, 2, \ldots, m - 1. \tag{A.8}$$

For $\widetilde{Q}_0(m; m) = (\sqrt{m})^{-1}\sum_{r=1}^{m}\widetilde{R}_r^r(m)$, the following relation is derived:

$$[\widetilde{R}_q^p(m), \ \widetilde{Q}_0(m; m)] = 0 \qquad \text{for any } p \text{ and } q. \tag{A.9}$$

The operator $\widetilde{Q}_0(m; r)$ is essentially equivalent to $\widetilde{R}_0(m; r)$. The operator $\widetilde{Q}_0(m; m)$ plays a role similar to the total fermion number $\widetilde{N}(n)$ in the $SU(n)$ Lipkin model. Thus, we know that the case with $n = 2m + 1$ contains the $SU(m)$ algebra in the $SU(n)$ Lipkin model.

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