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A possible framework of the Lipkin model obeying the $SU(n)$ algebra in arbitrary fermion number. I: The $SU(2)$ algebras extended from the conventional fermion pair and determination of the minimum weight states

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The minimum weight states of the Lipkin model consisting of n single-particle levels and obeying the $SU(n)$ algebra are investigated systematically. The basic idea is to use the $SU(2)$ algebra, which is independent of the $SU(n)$ algebra. This idea has already been presented by the present authors in the case of the conventional Lipkin model consisting of two single-particle levels and obeying the $SU(2)$ algebra. If this idea is followed, the minimum weight states are determined for any fermion number appropriately occupying n single-particle levels. Naturally, the conventional minimum weight state is included: all fermions occupy energetically the lowest single-particle level in the absence of interaction. The cases $n = 2$, 3, 4, and 5 are discussed in some detail.

Subject Index D50

1. Introduction

In 1965, in the early stages of the study of nuclear many-body theories, Lipkin, Meshkov, and Glick proposed a schematic model for understanding the microscopic structure of nuclear collective vibration [1]. Hereafter, we will call it the Lipkin model. Naturally, it was an up-to-date problem in those days. The Lipkin model treats many-fermion systems consisting of two single-particle levels with the same degeneracy as each other. In this paper, the degeneracy is denoted as 2Ω, which is a positive even number. For this model, we can construct the $SU(2)$ algebra in terms of certain bilinear forms in single-particle fermion operators under the condition that the total fermion number operator commutes with the $SU(2)$ generators. The Hamiltonian adopted in this model is expressed as a function of these $SU(2)$ generators. Concerning the total fermion number N, the simplest case may be the following: In the absence of interaction, all fermions fully occupy an energetically lower single-particle level, i.e., $N = 2\Omega$. Following the review article by Klein and Marshalek [2], we call this case a "closed-shell" system. Conventionally, only this case has been investigated. With the aid

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of this model, we are able to obtain a schematic understanding of collective vibrational states of the "closed-shell" system in terms of superposition of particle–hole pair excitations. In this case, it is easy to define the particle and the hole operators.

As a natural generalization of the Lipkin model, Li, Klein, and Dreizler [3] and Meshkov [4] first investigated the model consisting of three single-particle levels. Needless to say, this model is treated in the frame of the $SU(3)$ algebra. Further, the generalization to the case of n single-particle levels was performed mainly by Okubo [5] and Klein [6]. The degeneracy of each level is also equal to 2Ω. The mathematical framework in this case is given by the $SU(n)$ algebra with the condition that the total fermion number operator commutes with the $SU(n)$ generators. Needless to say, the Hamiltonian should be expressed as a function of the $SU(n)$ generators. Hereafter, we will call it the $SU(n)$ Lipkin model. Including the case $n \geq 3$, also only the "closed-shell" system, i.e., $N = 2\Omega$, has been investigated.

We guess that there exist two reasons why only the case $N = 2\Omega$ has been investigated. One of the reasons may be the following: The Lipkin model aims at describing the particle–hole pair type collective vibration and its ideal form may be expected to be realized in this case; it may thus not be necessary to investigate any case except for the "closed-shell" system. The second is related to the minimum weight state. The Lipkin model is a kind of algebraic model. Therefore, in order to complete the description of the model, the first task is to determine the minimum weight states. The "closed-shell" system corresponds to the simplest minimum weight state, which enables us to formulate various results of the Lipkin model quite easily. However, in the case of the $SU(2)$ Lipkin model, recently, the present authors proposed an idea [7]. Under this idea, the minimum weight states can be determined in the concrete form for the case of any fermion number. The prototype new boson realization of the $SU(2)$ algebra in the Lipkin model used in [7] can be found in [8]. This idea suggests that we may know the concrete forms of the minimum weight states of the $SU(n)$ Lipkin model for any fermion number. This problem will be discussed in this paper (I). However, even if the minimum weight state can be determined, we still have a problem to be solved. In the $SU(2)$ Lipkin model, the orthogonal set built on a chosen minimum weight state can easily be obtained by operating the raising operator successively on the minimum weight state. In the case of the $SU(n)$ Lipkin model, formally, there exist too many generators which play a role similar to that of the raising operator in the $SU(2)$ Lipkin model. Therefore, in order to make the $SU(n)$ Lipkin model workable, we must present any idea for the operators, the role of which is similar to that of the $SU(2)$ Lipkin model, i.e., the raising operator. This problem will be discussed in the next paper (II).

The main aim of this paper is to present concrete forms of the minimum weight states for any fermion number in the $SU(n)$ Lipkin model, including the "closed-shell" system. The preliminary argument was performed in the recent paper by the present authors for the $SU(2)$ Lipkin model [7]. In this argument, a certain $SU(2)$ algebra which is independent of the $SU(2)$ algebra in the Lipkin model plays a central role. We called it the auxiliary $SU(2)$ algebra. The orthogonal sets obtained under this algebra give us the minimum weight states of the $SU(2)$ Lipkin model. We extend this idea to the $SU(n)$ Lipkin model. The condition that the auxiliary $SU(2)$ algebra is independent of the $SU(n)$ algebra in the Lipkin model is formulated in the commutation relation

$$[\text{ any auxiliary } SU(2) \text{ generator },\ \text{any } SU(n) \text{ generator in the Lipkin model }] = 0. \quad (1.1)$$

To construct this auxiliary algebra, the raising operator in the $SU(2)$ algebra can be expressed in a certain form with the nth degree for the fermion creation operators and the Clifford numbers, unfamiliar to nuclear theory. The minimum weight states of the $SU(n)$ Lipkin model are given in terms of the orthogonal sets of the auxiliary $SU(2)$ algebra. In this paper, the terminology of the "closed-shell" system was used for the case in which, in the absence of interaction, all fermions occupy fully energetically the lowest single-particle level, i.e., $N = 2\Omega$. However, in order to formulate the "closed-shell" system rigorously, not only the condition $N = 2\Omega$ but also other conditions are necessary, for example, in the case of the $SU(2)$ Lipkin model, $s = \Omega$ (s: the magnitude of the $SU(2)$ spin for this model).

Recently, the excited state quantum phase transitions have drawn the attention in the field of nuclear many-body problems [9]. In particular, the Lipkin model is one of the important solvable models to understand the quantum phase transitions [10]. To give a possible framework of the $SU(n)$ Lipkin model may also be useful in order to investigate the physics of the excited state quantum phase transitions.

In next section, the $SU(n)$ Lipkin model is recapitulated and the condition governing the minimum weight states is given. In Sect. 3, the $SU(2)$ algebra auxiliary to the $SU(n)$ Lipkin model is formulated under the condition that any of the $SU(2)$ generators commutes with any of the $SU(n)$ generators. The three generators are expressed as functions of single-particle fermion operators. To obtain the expressions, the Clifford number may be necessary. In Sect. 4, formal aspects of the minimum weight states of the $SU(2)$ and $SU(3)$ Lipkin models are discussed. Section 5 is devoted to presenting the general forms of the minimum weight states concretely in the case of the $SU(n)$ Lipkin model. Finally, in Sect. 6, the minimum weight states for the $SU(n)$ Lipkin model in the cases $n = 2,\ 3,\ 4,$ and 5 are given in a form slightly different from that presented in Sect. 5, which will be useful for the discussion in (II).

2. The $SU(n)$ algebra in the Lipkin model

The many-fermion model discussed in this paper consists of n single-particle levels, the degeneracies of which are equal to $2\Omega = 2j + 1$ (j; half-integer). The single-particle states are specified by the quantum numbers (p, jm). Here, p and m are given by $p = 0, 1, 2, \ldots, n-1$ and $m = -j, -j+1, \ldots, j-1, j$, respectively. Hereafter, we omit the quantum number j. Following the order $p = 0 < p = 1 < \cdots < p = n-1$, the levels becomes higher. The level $p = 0$ is the lowest. The fermion operators are denoted by $(\tilde{c}_{p,m},\ \tilde{c}^*_{p,m})$ and, then, the total fermion number operator $\widetilde{N}(n)$ for the case n can be expressed as

$$\widetilde{N}(n) = \sum_{p=0}^{n-1} \sum_{m=-j}^{j} \tilde{c}^*_{p,m}\tilde{c}_{p,m}. \tag{2.1}$$

With the use of the above fermion operators, we can define the following operators for $p, q = 1, 2, \ldots, n-1$:

$$\widetilde{S}^p(n) = \sum_m \tilde{c}^*_{p,m}\tilde{c}_{0,m}, \qquad \widetilde{S}_p(n) = \sum_m \tilde{c}^*_{0,m}\tilde{c}_{p,m} \qquad \left(\widetilde{S}_p(n)^* = \widetilde{S}^p(n)\right), \tag{2.2a}$$

$$\widetilde{S}^p_q(n) = \sum_m \tilde{c}^*_{p,m}\tilde{c}_{q,m} - \delta_{pq}\sum_m \tilde{c}^*_{0,m}\tilde{c}_{0,m} \qquad \left(\widetilde{S}^q_p(n)^* = \widetilde{S}^p_q(n)\right). \tag{2.2b}$$

The commutation relations are given in the form

$$[\,\widetilde{S}^p(n),\ \widetilde{S}_q(n)\,] = \widetilde{S}^p_q(n), \tag{2.3a}$$

$$[\,\widetilde{S}^p_q(n),\ \widetilde{S}^r(n)\,] = \delta_{qr}\widetilde{S}^p(n) + \delta_{pq}\widetilde{S}^r(n), \tag{2.3b}$$

$$[\,\widetilde{S}^p_q(n),\ \widetilde{S}^s_r(n)\,] = \delta_{qs}\widetilde{S}^p_r(n) - \delta_{pr}\widetilde{S}^s_q(n). \tag{2.3c}$$

In relation (2.3), we can see that the operators (2.2) obey the $SU(n)$ algebra. The simplest Casimir operator, $\widetilde{\Gamma}_{SU(n)}$, is given as

$$\widetilde{\Gamma}_{SU(n)} = \frac{1}{2}\left[\sum_{p=1}^{n-1}\left(\widetilde{S}^p(n)\widetilde{S}_p(n) + \widetilde{S}_p(n)\widetilde{S}^p(n)\right) + \sum_{p,q=1}^{n-1}\widetilde{S}^p_q(n)\widetilde{S}^q_p(n) - \frac{1}{n}\left(\sum_{p=1}^{n-1}\widetilde{S}^p_p(n)\right)^2\right]. \tag{2.4}$$

The operators $\widetilde{\Gamma}_{SU(n)}$ and $\widetilde{N}(n)$ satisfy

$$[\,\widetilde{\Gamma}_{SU(n)} \text{ and } \widetilde{N}(n),\ \text{any of the operators (2.2)}\,] = 0. \tag{2.5}$$

Further, it should be noted that $\widetilde{N}(n)$ cannot be expressed in terms of the above $SU(n)$ generators.

For the above $SU(n)$ algebra, we can select a Hamiltonian

$$\widetilde{H}(n) = \widetilde{H}_0(n) + \widetilde{H}_1(n). \tag{2.6}$$

Here, $\widetilde{H}_0(n)$ is the Hamiltonian of individual levels with energies ε_p, for which we set up

$$\sum_{p=0}^{n-1}\varepsilon_p = 0, \qquad \varepsilon_0 \le \varepsilon_1 \le \cdots \le \varepsilon_{n-1}. \tag{2.7}$$

Then, $\widetilde{H}_0(n)$ can be expressed as

$$\widetilde{H}_0(n) = \sum_{p=0}^{n-1}\varepsilon_p\widetilde{N}_p(n) = \sum_{p=1}^{n-1}\varepsilon_p\widetilde{S}^p_p(n). \tag{2.8}$$

The part $\widetilde{H}_1(n)$ is an interaction term chosen, for illustration only, in the form

$$\widetilde{H}_1(n) = -G\sum_{p=1}^{n-1}\left[\left(\widetilde{S}^p(n)\right)^2 + \left(\widetilde{S}_p(n)\right)^2\right]. \tag{2.9}$$

Here, $G(>0)$ denotes the coupling constant. The above Hamiltonian can be found in Ref. [2] with different notation. We call the above many-fermion system the $SU(n)$ Lipkin model. The Hamiltonian $\widetilde{H}(n)$ obeys

$$\left[\,\widetilde{\Gamma}_{SU(n)} \text{ and } \widetilde{N}(n),\ \widetilde{H}(n)\,\right] = 0. \tag{2.10}$$

The cases $n = 2$ and 3 reduce to the Hamiltonians of the $SU(2)$ and the $SU(3)$ Lipkin model, which have been discussed in various problems [2].

For studies of any many-fermion system, implicitly or explicitly, we must prepare orthogonal sets for the system under investigation. The standard idea for treating the present model may be, first, to prepare an orthogonal set related by a chosen minimum weight state. The set may be constructed by operating the generators $\widetilde{S}^p(n)$ $(p = 1, 2, \ldots, n-1)$ and $\widetilde{S}^p_q(n)$ $(p > q = 1, 2, \ldots, n-2)$ *appropriately* on the minimum weight state, which we denote $|\min(n)\rangle$. The state $|\min(n)\rangle$ obeys the conditions

$$\widetilde{N}(n)|\min(n)\rangle = N_{n-1}|\min(n)\rangle, \tag{2.11}$$

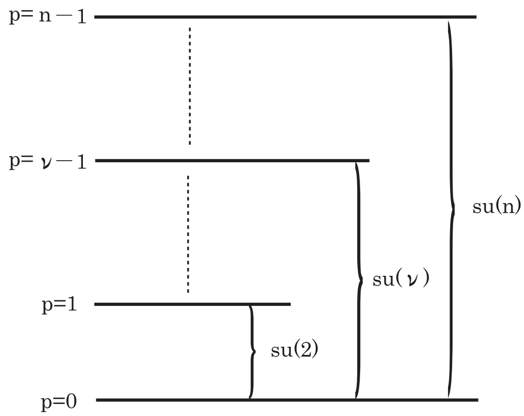


Fig. 1. The single-particle levels for the $SU(n)$ Lipkin models depicted schematically.

$$\widetilde{S}_p(n)|\min(n)\rangle = 0 \qquad (p = 1, 2, \ldots, n-1), \tag{2.12a}$$

$$\widetilde{S}_p^q(n)|\min(n)\rangle = 0 \qquad (p > q = 1, 2, \ldots, n-2) \tag{2.12b}$$

$$\widetilde{S}_p^p(n)|\min(n)\rangle = s_p(n)|\min(n)\rangle \qquad (p = 1, 2, \ldots, n-1). \tag{2.13}$$

Conventionally, for $|\min(n)\rangle$, a "closed-shell" system has been investigated:

$$N_{n-1} = 2\Omega, \qquad s_p(n) = -2\Omega \qquad (p = 1, 2, \ldots, n-1). \tag{2.14}$$

The above teaches us that the level $p = 0$ is fully occupied and the levels $p = 1, 2, \ldots, n-1$ are vacant. However, even if the treatment is restricted to the "closed-shell" system, there exist many "closed-shell" systems in the case $n \geq 4$; for example, the levels $p = 0$ and 1 are fully occupied and the other vacant:

$$N_{n-1} = 4\Omega, \qquad s_{p=1}(n) = 0, \qquad s_p(n) = -2\Omega \qquad (p = 2, 3, \ldots, n-1). \tag{2.15}$$

Including such "closed-shell" systems, it may be interesting to investigate the case with arbitrary fermion number, i.e., $0 \leq N_{n-1} \leq 2n\Omega$. Further, for constructing the orthogonal set built on $|\min(n)\rangle$, the appropriate choice of the operators as functions of $\widetilde{S}^p(n)$ $(p = 1, 2, \ldots, n-1)$ and $\widetilde{S}_q^p(n)$ $(p > q = 1, 2, \ldots, n-2)$ is inevitable. The simplest examples are given by $\widetilde{S}^p(n)|\min(n)\rangle$ and $\widetilde{S}_q^p(n)|\min(n)\rangle$. However, $\widetilde{S}_q^p(n)\widetilde{S}^q(n)|\min(n)\rangle$ and $\widetilde{S}^q(n)\widetilde{S}_q^p(n)|\min(n)\rangle$ are not independent of each other, because of the relation $[\ \widetilde{S}_q^p(n),\ \widetilde{S}^q(n)\ ] = \widetilde{S}^p(n)$. We call the appropriately chosen operators the "building blocks." The above argument tells us that, as was mentioned in Sect. 1, we have two tasks for formulating the present model: (1) to determine the minimum weight state, and (2) to construct the building blocks. Although these two are interrelated, the concrete contents are completely independent of each other. Therefore, after discussing task (1) in (I), we will consider task (2) in (II).

The main aim of this paper is to present an idea, under which the minimum weight states of the $SU(n)$ Lipkin model are systematically constructed. In order to make our idea understandable, we show the single-particle level scheme in Fig. 1. Let $|\min(\nu)\rangle$ denote a possible candidate of the minimum weight state of the $SU(\nu)$ Lipkin model for $2 \leq \nu \leq n$. We set up the following relations for $|\min(\nu)\rangle$:

$$\widetilde{N}(\nu)|\min(\nu)\rangle = N_{\nu-1}|\min(\nu)\rangle, \tag{2.16}$$

$$\widetilde{S}_p(\nu)|\min(\nu)\rangle = 0 \quad (p = 1, 2, \ldots, \nu - 1), \tag{2.17a}$$

$$\widetilde{S}_p^q(\nu)|\min(\nu)\rangle = 0 \quad (p > q = 1, 2, \ldots, \nu - 2), \tag{2.17b}$$

$$\widetilde{S}_p^p(\nu)|\min(\nu)\rangle = (\gamma_{\nu-1}(p) - \gamma_{\nu-1}(0))|\min(\nu)\rangle \quad (p = 1, 2, \ldots, \nu - 1). \tag{2.18}$$

Here, $(\gamma_{\nu-1}(p) - \gamma_{\nu-1}(0))$ is given through the relation

$$\sum_m \tilde{c}^*_{p,m}\tilde{c}_{p,m}|\min(\nu)\rangle = \gamma_{\nu-1}(p)|\min(\nu)\rangle \quad (p = 0, 1, \ldots, \nu - 1). \tag{2.19}$$

The total fermion number $N_{\nu-1}$ is expressed as

$$N_{\nu-1} = \sum_{p=0}^{\nu-1} \gamma_{\nu-1}(p). \tag{2.20}$$

It may be necessary to give some comment on the relations (2.18)–(2.20). The definitions of $\widetilde{N}(\nu)$ and $\widetilde{S}_p^p(\nu)$ shown in the relations (2.1) and (2.2b), respectively, for the case $n = \nu$ are rewritten in the form

$$\sum_m \tilde{c}^*_{p,m}\tilde{c}_{p,m} = \begin{cases} \dfrac{1}{\nu}\left(\widetilde{N}(\nu) - \displaystyle\sum_{q=1}^{\nu-1}\widetilde{S}_q^q(\nu)\right) & (p = 0), \quad (2.21a) \\ \widetilde{S}_p^p(\nu) + \dfrac{1}{\nu}\left(\widetilde{N}(\nu) - \displaystyle\sum_{q=1}^{\nu-1}\widetilde{S}_q^q(\nu)\right) & (p = 1, 2, \ldots, \nu - 1). \quad (2.21b) \end{cases}$$

The relation (2.21) tells us the following: Since the state $|\min(\nu)\rangle$ is regarded as the eigenstate of $\widetilde{N}(\nu)$ and $\widetilde{S}_p^p(\nu)$, $|\min(\nu)\rangle$ should also be the eigenstate of $\sum_m \tilde{c}^*_{0,m}\tilde{c}_{0,m}$ and $\sum_m \tilde{c}^*_{p,m}\tilde{c}_{p,m}$. Then, the relation (2.19) may be permitted to be set up, and the relations (2.18) and (2.20) are obtained.

The relations (2.16)–(2.20) are set up for the range $2 \leq \nu \leq n$. However, it may be convenient for later arguments to add the point $\nu = 1$ to $2 \leq \nu \leq n$. Judging from Fig. 1, it may be natural to consider that the case $\nu = 1$ may be restricted only to $p = 0$. Then, in this case, the relations (2.17) and (2.18) are meaningless and the relations (2.16) and (2.19) may be meaningful:

$$\widetilde{N}(1)|\min(1)\rangle = N_0|\min(1)\rangle = \gamma_0(0)|\min(1)\rangle. \tag{2.22a}$$

Here, $\widetilde{N}(1)$ is given by the relation (2.1) for $n = 1$ in the form

$$\widetilde{N}(1) = \sum_m \tilde{c}^*_{0,m}\tilde{c}_{0,m}. \tag{2.22b}$$

Let $|\min(\nu)\rangle$ be obtained. Then, we can show that $|\min(\nu)\rangle$ for $\nu = 2, 3, \ldots, n$ satisfies the relation

$$\widetilde{N}(n)|\min(\nu)\rangle = N_{\nu-1}|\min(\nu)\rangle, \tag{2.23}$$

$$\widetilde{S}_p(n)|\min(\nu)\rangle = 0 \quad (p = 1, 2, \ldots, n - 1), \tag{2.24a}$$

$$\widetilde{S}_p^q(n)|\min(\nu)\rangle = 0 \quad (p > q = 1, 2, \ldots, n - 2), \tag{2.24b}$$

$$\widetilde{S}_p^p(n)|\min(\nu)\rangle = \begin{cases} (\gamma_{\nu-1}(p) - \gamma_{\nu-1}(0))|\min(\nu)\rangle & (p = 1, 2, \ldots, \nu - 1)), \quad (2.25a) \\ -\gamma_{\nu-1}(0)|\min(\nu)\rangle & (p = \nu, \nu + 1, \ldots, n - 1). \quad (2.25b) \end{cases}$$

The reason is very simple. Since any fermion does not occupy the single-particle levels $p = \nu, \nu + 1, \ldots, n-1$, we have

$$\tilde{c}_{p,m}|\min(\nu)\rangle = 0 \quad (p = \nu, \nu+1, \ldots, n-1). \tag{2.26}$$

The relations (2.23)–(2.25) teach us that $|\min(\nu)\rangle$ as the solution of Eqs. (2.16)–(2.18) is also the minimum weight state of the $SU(n)$ Lipkin model. In the next section, we will discuss the $SU(2)$ algebra $(\widetilde{\Lambda}_{\pm,0}(n))$, which plays a central role for obtaining the state $|\min(\nu)\rangle$.

3. The $SU(2)$ algebra auxiliary to the $SU(n)$ Lipkin model

As mentioned in Sect. 1, an idea preliminary to the present one has already been shown in our recent paper for the case of the $SU(2)$ Lipkin model [7]. The basic idea is to introduce the $SU(2)$ algebra $(\widetilde{\Lambda}_{\pm,0}(2))$, which is characterized by the commutation relation

$$\left[\text{any of } \widetilde{\Lambda}_{\pm,0}(2), \text{ any of the } SU(2) \text{ generators } \left(\widetilde{S}^1(2), \widetilde{S}_1(2), \widetilde{S}_1^1(2)\right)\right] = 0. \tag{3.1}$$

The explicit forms are as follows:

$$\widetilde{\Lambda}_+(2) = \sum_m \tilde{c}^*_{1,m}\tilde{c}^*_{0,m}, \qquad \widetilde{\Lambda}_-(2) = \sum_m \tilde{c}_{0,m}\tilde{c}_{1,m}, \tag{3.2a}$$

$$\widetilde{\Lambda}_0(2) = \frac{1}{2}\sum_m (\tilde{c}^*_{1,m}\tilde{c}_{1,m} + \tilde{c}^*_{0,m}\tilde{c}_{0,m}) - \Omega \left(= \frac{1}{2}\widetilde{N}(2) - \Omega\right). \tag{3.2b}$$

It is easily verified that the expression (3.2) satisfies the condition (3.1) and obeys the $SU(2)$ algebra:

$$[\,\widetilde{\Lambda}_+(2),\ \widetilde{\Lambda}_-(2)\,] = 2\widetilde{\Lambda}_0(2), \qquad [\,\widetilde{\Lambda}_0(2),\ \widetilde{\Lambda}_\pm(2)\,] = \pm\widetilde{\Lambda}_\pm(2). \tag{3.3}$$

In our idea, $(\widetilde{\Lambda}_{\pm,0}(2))$ plays a central role in deriving the minimum weight state with arbitrary fermion number in the $SU(2)$ Lipkin model. Conventionally, only the case of the fermion number 2Ω has been treated, i.e., the "closed-shell" system. In Sect. 4, to illustrate our idea, we will discuss how $(\widetilde{\Lambda}_{\pm,0}(2))$ is used in our present problem including the case of the $SU(3)$ Lipkin model. In the form similar to the relation (3.2), we can give the $SU(2)$ algebra $(\widetilde{\Lambda}_{\pm,0}(n))$ which is independent of the $SU(n)$ Lipkin model:

$$\left[\text{ any of } \widetilde{\Lambda}_{\pm,0}(n), \text{ any of } \left(\widetilde{S}^p(n), \widetilde{S}_p(n), \widetilde{S}^p_q(n)\right)\right] = 0. \tag{3.4}$$

To construct $(\widetilde{\Lambda}_{\pm,0}(n))$, first, we must have a preliminary argument. We know that a system composed of one kind of fermion is regarded as a single $SU(2)$ spin system with the magnitude $1/2$. Through the following commutation relation, we can understand this point:

$$[\,\tilde{c}^*,\ \tilde{c}\,] = 2\left(\tilde{c}^*\tilde{c} - \frac{1}{2}\right), \qquad \left[\,\tilde{c}^*\tilde{c} - \frac{1}{2},\ \tilde{c}^*\,\right] = \tilde{c}^*, \qquad (\tilde{c}^*)^2 = 0. \tag{3.5}$$

Here, $(\tilde{c}^*, \tilde{c})$ denotes the fermion operator obeying the anti-commutation relation

$$\{\,\tilde{c}^*,\ \tilde{c}\,\} = 1, \qquad \{\,\tilde{c}^*,\ \tilde{c}^*\,\} = 0. \tag{3.6}$$

The anti-commutation relation (3.6) leads us to the relation (3.5). The fermion operators $\tilde{c}^*$ and $\tilde{c}$ play the roles of the raising and the lowering operators, respectively. However, the form (3.5) cannot be straightforwardly translated into the case of a many-fermion system, for example, the system specified by $p = 0$ in this paper:

$$\{ \tilde{c}^*_{0,m},\ \tilde{c}_{0,\mu} \} = \delta_{m\mu}, \qquad \{ \tilde{c}^*_{0,m},\ \tilde{c}^*_{0,\mu} \} = 0, \quad \text{i.e.} \quad (\tilde{c}_{0,m})^2 = 0. \tag{3.7}$$

The first relation of (3.7) is rewritten to

$$[\tilde{c}^*_{0,m},\ \tilde{c}_{0,\mu}] = 2\left(\tilde{c}^*_{0,m}\tilde{c}_{0,\mu} - \frac{1}{2}\delta_{m\mu}\right). \tag{3.8}$$

The form (3.8) suggests that it may be impossible to regard $\tilde{c}^*_{0,m}$ as the raising operator of a $SU(2)$ spin system as it stands.

Let us discuss a possible idea for the above problem. Under this idea, the present many-fermion system can be regarded as that composed of independent 2Ω $SU(2)$ spins. Each is specified by m and its magnitude is equal to $1/2$. This idea is realized through introducing the Clifford numbers e_m $(m = -j, -j+1, \ldots, j-1, j)$, which obey the condition

$$\begin{aligned} &e_m e_\mu + e_\mu e_m = 0 \quad \text{for} \quad m \neq \mu, \quad (e_m)^2 = 1, \qquad \text{i.e.,}\ \{ e_m,\ e_\mu \} = 2\delta_{m\mu}, \\ &[e_m,\ \tilde{c}^*_{0,\mu} \text{ and } \tilde{c}_{0,\mu}] = 0. \end{aligned} \tag{3.9}$$

Of course, e_m commutes with the fermion operators. With the use of e_m, we define the following operators:

$$\tilde{d}^*_{0,m} = e_m \tilde{c}^*_{0,m}, \qquad \tilde{d}_{0,m} = e_m \tilde{c}_{0,m}. \tag{3.10}$$

With the aid of the anti-commutation relation (3.7) and the property of the Clifford number (3.9), we can derive the following relation[1] for $(\tilde{d}^*_{0,m}, \tilde{d}_{0,m})$:

$$[\tilde{d}^*_{0,m},\ \tilde{d}^*_{0,\mu}] = 0, \qquad (\tilde{d}^*_{0,m})^2 = 0, \tag{3.11a}$$

$$[\tilde{d}^*_{0,m},\ \tilde{d}_{0,\mu}] = \delta_{m\mu} \cdot 2\left(\tilde{d}^*_{0,m}\tilde{d}_{0,m} - \frac{1}{2}\right), \tag{3.11b}$$

$$\left[\tilde{d}^*_{0,m}\tilde{d}_{0,m} - \frac{1}{2},\ \tilde{d}^*_{0,\mu} \right] = \delta_{m\mu} \cdot \tilde{d}^*_{0,\mu}. \tag{3.11c}$$

In contrast to the form (3.8), we can see that the symbol $\delta_{m\mu}$ is attached to both the terms on the right-hand side of the relation (3.11b). Therefore, the relation (3.11) suggests that the present many-fermion system consists of 2Ω $SU(2)$ spins which are independent of one other, and the generators of

[1] Equation (3.11b) can be derived through the following process:

$$\begin{aligned} [\tilde{d}^*_{0,m},\ \tilde{d}_{0,\mu}] &= e_m\tilde{c}^*_{0,m} \cdot e_\mu\tilde{c}_{0,\mu} - e_\mu\tilde{c}_{0,\mu} \cdot e_m\tilde{c}^*_{0,m} \\ &= e_m e_\mu \cdot \tilde{c}^*_{0,m}\tilde{c}_{0,\mu} - e_\mu e_m \cdot (\delta_{m\mu} - \tilde{c}^*_{0,m}\tilde{c}_{0,\mu}) \\ &= \{ e_m,\ e_\mu \} \cdot \tilde{c}^*_{0,m}\tilde{c}_{0,\mu} - e_m^2 \cdot \delta_{m\mu} = \delta_{m\mu} \cdot 2\left(\tilde{c}^*_{0,m}\tilde{c}_{0,\mu} - \tfrac{1}{2}\right) \\ &= \delta_{m\mu} \cdot 2\left(\tilde{d}^*_{0,m}\tilde{d}_{0,m} - \tfrac{1}{2}\right). \end{aligned}$$

the mth spin are given by $(\tilde{d}^*_{0,m}, \tilde{d}_{0,m}, \tilde{d}^*_{0,m}\tilde{d}_{0,m} - 1/2)$. The total spin of the present system, $(\widetilde{\Lambda}_{\pm,0}(1))$, can be expressed in the form

$$\widetilde{\Lambda}_+(1) = \sum_m \tilde{d}^*_{0,m} \left(= \sum_m e_m \tilde{c}^*_{0,m}\right), \qquad \widetilde{\Lambda}_-(1) = \sum_m \tilde{d}_{0,m} \left(= \sum_m e_m \tilde{c}_{0,m}\right), \tag{3.12a}$$

$$\widetilde{\Lambda}_0(1) = \sum_m \left(\tilde{d}^*_{0,m}\tilde{d}_{0,m} - \frac{1}{2}\right) \left(= \sum_m \tilde{c}^*_{0,m}\tilde{c}_{0,m} - \Omega = \widetilde{N}(1) - \Omega\right) \quad (e_m^2 = 1). \tag{3.12b}$$

Of course, they obey the $SU(2)$ algebra:

$$[\,\widetilde{\Lambda}_+(1),\ \widetilde{\Lambda}_-(1)\,] = 2\widetilde{\Lambda}_0(1), \qquad [\,\widetilde{\Lambda}_0(1),\ \widetilde{\Lambda}_\pm(1)\,] = \pm\widetilde{\Lambda}_\pm(1). \tag{3.13}$$

We can treat the eigenvalue problem of $(\widetilde{\Lambda}_{\pm,0}(1))$, which will be discussed in Sect. 4 in reference to the $SU(2)$ and $SU(3)$ Lipkin model. The $SU(2)$ algebra $(\tilde{\Lambda}_{\pm,0}(2))$ given in the relation (3.2) is expressed as

$$\widetilde{\Lambda}_+(2) = \sum_m \tilde{d}^*_{1,m}\tilde{d}^*_{0,m}, \qquad \widetilde{\Lambda}_-(2) = \sum_m \tilde{d}_{0,m}\tilde{d}_{1,m}, \tag{3.14a}$$

$$\widetilde{\Lambda}_0(2) = \frac{1}{2}\sum_m (\tilde{d}^*_{1,m}\tilde{d}_{1,m} + \tilde{d}^*_{0,m}\tilde{d}_{0,m}) - \Omega. \tag{3.14b}$$

Here, we used $(e_m)^2 = 1$, and $(\tilde{d}^*_{0,m}, \tilde{d}_{0,m})$ and $(\tilde{d}^*_{1,m}, \tilde{d}_{1,m})$ are given through

$$\tilde{d}^*_{p,m} = e_m\tilde{c}^*_{p,m}, \qquad \tilde{d}_{p,m} = e_m\tilde{c}_{p,m} \quad (p = 0, 1, \ldots, n-1). \tag{3.15}$$

The properties of the above operators are summarized as follows:[2]

$$\{\,\tilde{d}^*_{p,m},\ \tilde{d}^*_{q,\mu}\,\} = 0, \qquad \{\,\tilde{d}_{p,m},\ \tilde{d}^*_{q,\mu}\,\} = \delta_{pq} \quad \text{for} \quad m = \mu, \tag{3.15a}$$

$$[\,\tilde{d}^*_{p,m},\ \tilde{d}^*_{q,\mu}\,] = 0, \qquad [\,\tilde{d}_{p,m},\ \tilde{d}^*_{q,\mu}\,] = 0 \quad \text{for} \quad m \neq \mu. \tag{3.15b}$$

We are now able to give explicit forms for $\widetilde{\Lambda}_{\pm,0}(n)$. First, we define the following operators:

$$\tilde{d}^*_m(n) = \tilde{d}^*_{n-1,m}\tilde{d}^*_{n-2,m}\cdots\tilde{d}^*_{1,m}\tilde{d}^*_{0,m}, \tag{3.16}$$

i.e.,

$$\tilde{d}^*_m(n) = \begin{cases} \tilde{c}^*_{n-1,m}\tilde{c}^*_{n-2,m}\cdots\tilde{c}^*_{1,m}\tilde{c}^*_{0,m} & \text{for } n \text{ even } ((e_m)^n = 1), \quad (3.17a) \\ e_m\tilde{c}^*_{n-1,m}\tilde{c}^*_{n-2,m}\cdots\tilde{c}^*_{1,m}\tilde{c}^*_{0,m} & \text{for } n \text{ odd } ((e_m)^n = e_m). \quad (3.17b) \end{cases}$$

Clearly, $\tilde{d}^*_m(1) = \tilde{d}^*_{0,m}$ and $\tilde{d}^*_m(2) = \tilde{d}^*_{1,m}\tilde{d}^*_{0,m}$, which were used in the expressions (3.12) and (3.14), respectively. The operators $(\tilde{d}^*_m(n), \tilde{d}_m(n))$ satisfy the relation

$$\tilde{d}^*_m(n) = \tilde{d}^*_m(n)\cdot\tilde{d}_m(n)\cdot\tilde{d}^*_m(n), \tag{3.18a}$$

$$(\tilde{d}^*_m(n))^2 = 0. \tag{3.18b}$$

[2] The second of the relations in (3.15a) can be derived through the following process:

$$\begin{aligned}\{\,\tilde{d}_{p,m},\ \tilde{d}^*_{q,m}\,\} &= e_m\tilde{c}_{p,m}\cdot e_m\tilde{c}^*_{q,m} + e_m\tilde{c}^*_{q,m}\cdot e_m\tilde{c}_{p,m} \\ &= (e_m)^2\{\,\tilde{c}_{p,m},\ \tilde{c}^*_{q,m}\,\} = \delta_{pq}.\end{aligned}$$

The above two relations are compatible with each other. Further, we have

$$[\ \tilde{d}_m^*(n),\ \tilde{d}_\mu^*(n)\ ] = 0 \quad \text{for any combination of } (m,\mu), \tag{3.19a}$$

$$[\ \tilde{d}_m^*(n),\ \tilde{d}_\mu(n)\ ] = 0 \quad \text{for } m \neq \mu. \tag{3.19b}$$

Judging from the expressions (3.12) and (3.14), it may be natural to set up the following form for $(\widetilde{\Lambda}_{\pm,0}(n))$:

$$\widetilde{\Lambda}_+(n) = \sum_m \tilde{d}_m^*(n), \qquad \widetilde{\Lambda}_-(n) = \sum_m \tilde{d}_m(n), \tag{3.20a}$$

$$\widetilde{\Lambda}_0(n) = \frac{1}{2}\sum_m [\ \tilde{d}_m^*(n),\ \tilde{d}_m(n)\ ]. \tag{3.20b}$$

It should be noted that the $SU(2)$ algebra $(\widetilde{\Lambda}_{\pm,0}(n))$ is extended from the fermion pair for $p = 0$ and 1 $(\widetilde{\Lambda}_{\pm,0}(2))$. With the use of the relations (3.18) and (3.19), we can show that $\widetilde{\Lambda}_{\pm,0}(n)$ obey the $SU(2)$ algebra:

$$[\ \widetilde{\Lambda}_+(n),\ \widetilde{\Lambda}_-(n)\ ] = 2\widetilde{\Lambda}_0(n), \qquad [\ \widetilde{\Lambda}_0(n),\ \widetilde{\Lambda}_\pm(n)\ ] = \pm\widetilde{\Lambda}_\pm(n). \tag{3.21}$$

Next, we will give the proof of the commutation relation (3.4). For this, we express the $SU(n)$ generators (2.2) in the unified form

$$\widetilde{S}_\sigma^\rho(n) = \sum_m (\tilde{c}_{\rho,m}^*\tilde{c}_{\sigma,m} - \delta_{\rho\sigma}\tilde{c}_{0,m}^*\tilde{c}_{0,m}) \qquad (\rho,\sigma = 0,1,\ldots,n-2,n-1). \tag{3.22}$$

Of course, $\widetilde{S}_0^0(n) = 0$. On the other hand, picking up $\tilde{c}_{\rho,m}^*\tilde{c}_{\sigma,m}$, $\widetilde{\Lambda}_+(n)$ shown in the relation (3.20) with (3.17) can be factorized as follows:

$$\widetilde{\Lambda}_+(n) = \sum_m \widetilde{\Lambda}_m^{(+)}(n;\rho\sigma)\cdot\tilde{c}_{\rho,m}^*\tilde{c}_{\sigma,m}^*. \tag{3.23}$$

It should be noted that $\widetilde{\Lambda}_m^{(+)}(n;\rho\sigma)$ does not contain $\tilde{c}_{\rho,m}^*\tilde{c}_{\sigma,m}^*$. Then, for $\rho \neq \sigma$, we have

$$[\ \widetilde{\Lambda}_+(n),\ \widetilde{S}_\sigma^\rho(n)\ ] = \sum_m \widetilde{\Lambda}_m^{(+)}(n;\rho\sigma)[\ \tilde{c}_{\rho,m}^*\tilde{c}_{\sigma,m}^*,\ \tilde{c}_{\rho,m}^*\tilde{c}_{\sigma,m}\ ] = 0, \tag{3.24a}$$

$$[\ \widetilde{\Lambda}_-(n),\ \widetilde{S}_\sigma^\rho(n)\ ] = -[\ \widetilde{\Lambda}_+(n),\ \widetilde{S}_\rho^\sigma(n)\ ]^* = 0. \tag{3.24b}$$

For the case $\rho = \sigma$, we have

$$[\ \widetilde{\Lambda}_+(n),\ \widetilde{S}_\rho^\rho(n)\ ] = \sum_m \widetilde{\Lambda}_m^{(+)}(n;\rho\sigma=0)[\ \tilde{c}_{\rho,m}^*\tilde{c}_{0,m}^*,\ \tilde{c}_{\rho,m}^*\tilde{c}_{\rho,m} - \tilde{c}_{0,m}^*\tilde{c}_{0,m}\ ] = 0, \tag{3.25a}$$

$$[\ \widetilde{\Lambda}_-(n),\ \widetilde{S}_\rho^\rho(n)\ ] = -[\ \widetilde{\Lambda}_+(n),\ \widetilde{S}_\rho^\rho(n)\ ]^* = 0. \tag{3.25b}$$

The relation $\widetilde{\Lambda}_0(n) = [\ \widetilde{\Lambda}_+(n),\ \widetilde{\Lambda}_-(n)\ ]/2$ gives us

$$[\ \widetilde{\Lambda}_0(n),\ \widetilde{S}_\sigma^\rho(n)\ ] = 0. \tag{3.26}$$

In this way, we could show that the expression (3.20) satisfies the relation (3.4).

In the next section, the expressions of $\widetilde{\Lambda}_{\pm,0}(2)$ shown in the relation (3.14) and $\widetilde{\Lambda}_{\pm,0}(3)$ shown in the following play a central role:

$$\widetilde{\Lambda}_+(3) = \sum_m e_m\tilde{c}_{2,m}^*\tilde{c}_{1,m}^*\tilde{c}_{0,m}^*, \qquad \widetilde{\Lambda}_-(3) = \sum_m e_m\tilde{c}_{0,m}\tilde{c}_{1,m}\tilde{c}_{2,m}, \tag{3.27a}$$

$$\widetilde{\Lambda}_0(3) = \frac{1}{2}\sum_m (\tilde{c}^*_{2,m}\tilde{c}_{2,m} + \tilde{c}^*_{1,m}\tilde{c}_{1,m} + \tilde{c}^*_{0,m}\tilde{c}_{0,m})$$
$$- \frac{1}{2}\sum_m (\tilde{c}^*_{2,m}\tilde{c}_{2,m}\cdot\tilde{c}^*_{1,m}\tilde{c}_{1,m} + \tilde{c}^*_{1,m}\tilde{c}_{1,m}\cdot\tilde{c}^*_{0,m}\tilde{c}_{0,m} + \tilde{c}^*_{0,m}\tilde{c}_{0,m}\cdot\tilde{c}^*_{2,m}\tilde{c}_{2,m})$$
$$+ \sum_m \tilde{c}^*_{2,m}\tilde{c}_{2,m}\cdot\tilde{c}^*_{1,m}\tilde{c}_{1,m}\cdot\tilde{c}^*_{0,m}\tilde{c}_{0,m} - \Omega. \quad (3.27b)$$

4. The minimum weight states of the $SU(2)$ and $SU(3)$ Lipkin model

In order to illustrate our idea, let us start with the $SU(2)$ Lipkin model. We denote one of the states in which only the single-particle level $p = 0$ is occupied by N_0 fermions as $|N_0\rangle$:

$$\widetilde{N}(1)|N_0\rangle = N_0|N_0\rangle, \quad \text{i.e.,} \quad \widetilde{N}(2)|N_0\rangle = N_0|N_0\rangle. \quad (4.1)$$

Here, we omitted any quantum number which does not connect with the algebras under consideration. It is easily verified that $|N_0\rangle$ is a possible candidate of the minimum weight states of the $SU(2)$ Lipkin model:

$$\widetilde{S}_1(2)|N_0\rangle = 0, \qquad \widetilde{S}^1_1(2)|N_0\rangle = -N_0|N_0\rangle \quad (N_0 \geq 0). \quad (4.2)$$

Comparison of the relations (4.1) and (4.2) with (2.24), (2.15a), and (2.26) gives us, for the case $(n = 2,\ \nu = 1,\ p = 1)$:

$$|\min(1)\rangle = |N_0\rangle, \qquad \gamma_0(0) = N_0. \quad (4.3)$$

An example of $|N_0\rangle$ is presented in the appendix.

The state $|N_0\rangle$ is also the minimum weight state of the $SU(2)$ algebra $(\widetilde{\Lambda}_{\pm,0}(2))$:

$$\widetilde{\Lambda}_-(2)|N_0\rangle = 0, \quad (4.4a)$$
$$\widetilde{\Lambda}_0(2)|N_0\rangle = -\lambda(2)|N_0\rangle, \qquad \lambda(2) = \Omega - \frac{N_0}{2}. \quad (4.4b)$$

Therefore, by operating $\widetilde{\Lambda}_+(2)$ successively on $|N_0\rangle$, we are able to obtain the states orthogonal to $|N_0\rangle$ in the form

$$|N_1, N_0\rangle = \left(\widetilde{\Lambda}_+(2)\right)^{\frac{N_1-N_0}{2}}|N_0\rangle \quad (0 \leq N_0 \leq N_1). \quad (4.5)$$

The state $|N_1, N_0\rangle$ satisfies

$$\widetilde{N}(2)|N_1, N_0\rangle = N_1|N_1, N_0\rangle, \quad (4.6)$$
$$\widetilde{S}_1(2)|N_1, N_0\rangle = 0, \qquad \widetilde{S}^1_1(2)|N_1, N_0\rangle = -N_0|N_1, N_0\rangle, \quad (4.7)$$

$$\widetilde{\Lambda}_-(2)|N_1, N_0\rangle \neq 0, \quad (4.8a)$$
$$\widetilde{\Lambda}_0(2)|N_1, N_0\rangle = \lambda_0(2)|N_1, N_0\rangle, \qquad \lambda_0(2) = \frac{N_1 - N_0}{2} - \lambda(2). \quad (4.8b)$$

The state $|N_1, N_0\rangle$ is also the minimum weight state of the $SU(2)$ Lipkin model with the same property as that shown in the relation (4.2). But, it is not the minimum weight state of the $SU(2)$ algebra $(\widetilde{\Lambda}_{\pm,0}(2))$. For $|\min(2)\rangle = |N_1, N_0\rangle$, we obtain the following:

$$\gamma_1(0) = \frac{N_1 - N_0}{2} + N_0, \qquad \gamma_1(1) = \frac{N_1 - N_0}{2}. \quad (4.9a)$$

Inversely, we have

$$N_0 = \gamma_1(0) - \gamma_1(1), \qquad N_1 = \gamma_1(0) + \gamma_1(1). \tag{4.9b}$$

The above relations lead us to the inequalities

$$0 \le N_0 \le N_1, \qquad 0 \le \gamma_1(1) \le \gamma_1(0). \tag{4.10}$$

Since $\widetilde{\Lambda}_{\pm,0}(2)$ obey the $SU(2)$ algebra, the relations (4.4b) and (4.8b) give us the following inequalities:

$$0 \le \Omega - \frac{N_0}{2}, \quad \text{i.e.,} \quad 0 \le N_0 \le 2\Omega, \tag{4.11a}$$

$$-\left(\Omega - \frac{N_0}{2}\right) \le -\left(\Omega - \frac{N_1}{2}\right) \le \Omega - \frac{N_0}{2}, \quad \text{i.e.,} \quad 0 \le N_0 \le N_1 \le 4\Omega - N_0. \tag{4.11b}$$

Fermion numbers in the single-particle levels $p = 0$ and $p = 1$ are given in the relation (4.10) and, then, we have

$$0 \le \gamma_1(1) \le \gamma_1(0) \le 2\Omega. \tag{4.12}$$

Of course, if $N_1 = N_0$, $\gamma_0(1) = N_0$ and $\gamma_1(1) = 0$. The above is an outline of the $SU(2)$ Lipkin model based on the present idea and, needless to say, it is consistent with the result shown in our recent work. We were able to obtain the minimum weight states of the Lipkin model with any fermion numbers governed by the condition (4.12).

Next, we consider the minimum weight states of the $SU(3)$ Lipkin model. First, we pay attention to the state $|N_1, N_0\rangle$ shown in the relation (4.5), which satisfies

$$\widetilde{N}(3)|N_1, N_0\rangle = N_1|N_1, N_0\rangle, \tag{4.13}$$

$$\widetilde{S}_1(3)|N_1, N_0\rangle = \widetilde{S}_2(3)|N_1, N_0\rangle = \widetilde{S}_2^1(3)|N_1, N_0\rangle = 0, \tag{4.14}$$

$$\widetilde{S}_1^1(3)|N_1, N_0\rangle = -N_0|N_1, N_0\rangle, \qquad \widetilde{S}_2^2(3)|N_1, N_0\rangle = -\frac{1}{2}(N_1 + N_0)|N_1, N_0\rangle. \tag{4.15}$$

For the relation (4.14), we should note that $\widetilde{S}_1(3) = \widetilde{S}_1(2)$ and, further, $|N_1, N_0\rangle$ does not contain any fermion in the level $p = 2$, and $\widetilde{S}_2(3)$ and $\widetilde{S}_2^1(3)$ contain the annihilation operator in $p = 2$. Although $|N_1, N_0\rangle$ is not the minimum weight state of $(\widetilde{\Lambda}_{\pm,0}(2))$, it is the minimum weight state of $(\widetilde{\Lambda}_{\pm,0}(3))$:

$$\widetilde{\Lambda}_-(3)|N_1, N_0\rangle = 0, \tag{4.16a}$$

$$\widetilde{\Lambda}_0(3)|N_1, N_0\rangle = -\lambda(3)|N_1, N_0\rangle, \qquad \lambda(3) = \Omega - \frac{1}{2}\left(\frac{N_1 - N_0}{2} + N_0\right). \tag{4.16b}$$

If $N_1 = N_0$, $|N_0\rangle$ $(= |N_1 = N_0, N_0\rangle)$ is also the minimum weight state of the $SU(3)$ Lipkin model. This may be clear from the relations (4.13)–(4.16). Then, we introduce the state $|N_2, N_1, N_0\rangle$ in the form

$$|N_2, N_1, N_0\rangle = \left(\widetilde{\Lambda}_+(3)\right)^{\frac{N_2 - N_1}{3}} |N_1, N_0\rangle. \tag{4.17}$$

The state $|N_2, N_1, N_0\rangle$ satisfies

$$\widetilde{N}(3)|N_2, N_1, N_0\rangle = N_2|N_2, N_1, N_0\rangle, \tag{4.18}$$

$$\widetilde{S}_1(3)|N_2, N_1, N_0\rangle = \widetilde{S}_2(3)|N_2, N_1, N_0\rangle = \widetilde{S}_2^1(3)|N_2, N_1, N_0\rangle = 0, \tag{4.19}$$

$$\widetilde{S}_1^1(3)|N_2, N_1, N_0\rangle = -N_0|N_2, N_1, N_0\rangle, \;\; \widetilde{S}_2^2(3)|N_2, N_1, N_0\rangle = -\frac{1}{2}(N_1 + N_0)|N_2, N_1, N_0\rangle, \tag{4.20}$$

$$\widetilde{\Lambda}_-(3)|N_2, N_1, N_0\rangle \neq 0, \tag{4.21a}$$

$$\widetilde{\Lambda}_0(3)|N_2, N_1, N_0\rangle = \lambda_0(3)|N_2, N_1, N_0\rangle, \qquad \lambda_0(3) = \frac{N_2 - N_1}{3} - \lambda(3). \tag{4.21b}$$

The state $|N_2, N_1, N_0\rangle$ is also the minimum weight state of the $SU(3)$ Lipkin model with the same property as that shown in (4.14) and (4.15). But, it is not the minimum weight state of the $SU(2)$ algebra ($\widetilde{\Lambda}_{\pm,0}(3)$). For $|\min(3)\rangle = |N_2, N_1, N_0\rangle$, we obtain the following:

$$\begin{aligned} \gamma_2(0) &= \frac{N_2 - N_1}{3} + \frac{N_1 - N_0}{2} + N_0, \\ \gamma_2(1) &= \frac{N_2 - N_1}{3} + \frac{N_1 - N_0}{2}, \\ \gamma_2(2) &= \frac{N_2 - N_1}{3}. \end{aligned} \tag{4.22a}$$

Inversely, we have

$$N_0 = \gamma_2(0) - \gamma_2(1), \quad N_1 = \gamma_2(0) + \gamma_2(1) - 2\gamma_2(2), \quad N_2 = \gamma_2(0) + \gamma_2(1) + \gamma_2(2). \tag{4.22b}$$

The above relations lead us to

$$0 \leq N_0 \leq N_1 \leq N_2, \qquad 0 \leq \gamma_2(2) \leq \gamma_2(1) \leq \gamma_2(0). \tag{4.23}$$

The operators $\widetilde{\Lambda}_{\pm,0}(3)$ obey the $SU(2)$ algebra and, then, the relations (4.15) and (4.21b) lead us to the following inequalities:

$$0 \leq \Omega - \frac{N_1 + N_0}{4}, \tag{4.24a}$$

$$-\left(\Omega - \frac{N_1 + N_0}{4}\right) \leq -\left(\Omega - \frac{N_2 - N_1}{3} - \frac{N_1 + N_0}{4}\right) \leq \Omega - \frac{N_1 + N_0}{4}. \tag{4.24b}$$

The relations (4.24a) and (4.24b), together with the inequality in the relation (4.23), are rewritten as

$$0 \leq N_1 \leq 4\Omega - N_0, \tag{4.25a}$$

$$0 \leq N_0 \leq N_1 \leq N_2 \leq 6\Omega - \frac{1}{2}(N_1 + 3N_0). \tag{4.25b}$$

The relation (4.23) gives us

$$0 \leq \gamma_2(2) \leq \gamma_2(1) \leq \gamma_2(0) \leq 2\Omega. \tag{4.26}$$

Needless to say, $|N_0\rangle$ and $|N_1, N_0\rangle$ are also the minimum weight states of the $SU(3)$ Lipkin model. In the cases ($N_2 = N_1 = N_0$) and ($N_2 = N_1 > N_0$), $|N_2, N_1, N_0\rangle$ are reduced to $|N_0\rangle$ and $|N_1, N_0\rangle$, respectively.

5. The minimum weight states of the general case

In last section, we discussed the cases of the $SU(2)$ and $SU(3)$ Lipkin model. As given in the relations (4.4) and (4.16), $|N_0\rangle$ and $|N_1, N_0\rangle$ are the minimum weight states of $(\widetilde{\Lambda}_{\pm,0}(n))$ for $n = 2$ and 3, respectively. The example of $|N_0\rangle$ and the explicit form of $|N_1, N_0\rangle$ are shown in relations (A.7) and (4.5), respectively. These two forms suggest the following form:

$$|N_{n-2}, N_{n-3}, \ldots, N_1, N_0\rangle = \left(\widetilde{\Lambda}_+(n-1)\right)^{\frac{N_{n-2}-N_{n-3}}{n-1}} \cdot \left(\widetilde{\Lambda}_+(n-2)\right)^{\frac{N_{n-3}-N_{n-4}}{n-2}} \cdots \times \left(\widetilde{\Lambda}_+(2)\right)^{\frac{N_1-N_0}{2}} |N_0\rangle$$
$$= \prod_{\nu=2}^{n-1} \left(\widetilde{\Lambda}_+(\nu)\right)^{\frac{N_{\nu-1}-N_{\nu-2}}{\nu}} |N_0\rangle \quad (n \geq 3). \tag{5.1a}$$

If we adopt the form (A.7), the state (5.1a) can be expressed as

$$|N_{n-2}, N_{n-3}, \ldots, N_1, N_0\rangle = \prod_{\nu=1}^{n-1} \left(\widetilde{\Lambda}_+(\nu)\right)^{\frac{N_{\nu-1}-N_{\nu-2}}{\nu}} |N\rangle \quad \text{for} \quad N_{-1} = N \quad (n \geq 2). \tag{5.1b}$$

Hereafter, we will use only the form (5.1a). Therefore, our treatment is valid for $n \geq 3$. If the form (5.1) is accepted, the minimum weight state of the $SU(n)$ Lipkin model may be given as

$$|N_{n-1}, N_{n-2}, N_{n-3}, \ldots, N_1, N_0\rangle = \left(\widetilde{\Lambda}_+(n)\right)^{\frac{N_{n-1}-N_{n-2}}{n}} |N_{n-2}, N_{n-3}, \ldots, N_1, N_0\rangle. \tag{5.2}$$

First, let us prove the relation

$$\widetilde{\Lambda}_-(n)|N_{n-2}, N_{n-3}, \ldots, N_1, N_0\rangle = 0. \tag{5.3}$$

For this, some preliminary argument is necessary. For the case $\nu < n$, the operator $\tilde{d}_m(n)$ introduced in the relation (3.15) can be factorized into the form

$$\tilde{d}_m(n) = \tilde{d}_m(\nu) \cdot \tilde{\delta}_m(n, \nu), \tag{5.4}$$

$$\tilde{d}_m(\nu) = \tilde{d}_{0,m}\tilde{d}_{1,m} \cdots \tilde{d}_{\nu-1,m}, \tag{5.5a}$$

$$\tilde{\delta}_m(n, \nu) = \tilde{d}_{\nu,m}\tilde{d}_{\nu+1,m} \cdots \tilde{d}_{n-1,m}. \tag{5.5b}$$

The operator $(\tilde{\delta}_m^*(n, \nu), \tilde{\delta}_m(n, \nu))$ satisfies

$$[\, \tilde{\delta}_m^*(n, \nu),\ \tilde{d}_\mu^*(\nu')\,] = 0, \qquad [\, \tilde{\delta}_m(n, \nu),\ \tilde{d}_\mu^*(\nu')\,] = 0 \quad \text{for} \quad \nu' \leq \nu. \tag{5.6}$$

The relation (5.6) may be self-evident, because $(\tilde{\delta}_m^*(n, \nu), \tilde{\delta}_m(n, \nu))$ and $\tilde{d}_\mu^*$ are composed from operators which are different from each other. This can be seen in the relation (5.5). The operator $\widetilde{\Lambda}_-(n)$ is expressed as

$$\widetilde{\Lambda}_-(n) = \sum_m \tilde{d}_m(n) = \sum_m \tilde{d}_m(\nu) \cdot \tilde{\delta}_m(n, \nu). \tag{5.7}$$

Then, using relations (3.19) and (5.6), we have

$$[\, \widetilde{\Lambda}_-(n),\ \widetilde{\Lambda}_+(\nu)\,] = \sum_m [\, \tilde{d}_m(\nu),\ \tilde{d}_m^*(\nu)\,] \cdot \tilde{\delta}_m(n, \nu) \quad \text{for} \quad \nu < n, \tag{5.8a}$$

$$[\ \tilde{\delta}_m^*(n,\nu),\ \widetilde{\Lambda}_+(\nu')\ ] = 0, \qquad [\ \tilde{\delta}_m(n,\nu),\ \widetilde{\Lambda}_+(\nu')\ ] = 0 \quad \text{for} \quad \nu' \leq \nu. \tag{5.8b}$$

Successive use of the relation (5.8) and the condition $\widetilde{\Lambda}_-(n)|N_0\rangle = 0$ lead us to the relation (5.3).

Next, we consider that the state $|N_{n-2}, N_{n-3}, \ldots, N_1, N_0\rangle$ is the eigenstate of $\widetilde{\Lambda}_0(n)$, and its eigenvalue should be obtained. The relations (3.20b), (5.4), and (5.6) lead us to $\widetilde{\Lambda}_0(n)$ in the following form:

$$\begin{aligned}\widetilde{\Lambda}_0(n) = &-\frac{1}{2}\sum_m \tilde{d}_m(\nu)\tilde{d}_m^*(\nu)\\ &+\frac{1}{2}\Bigg(\sum_m \tilde{d}_m^*(\nu)\tilde{d}_m(\nu)\cdot\tilde{\delta}_m^*(n,\nu)\tilde{\delta}_m(n,\nu)\\ &\qquad + \tilde{d}_m(\nu)\tilde{d}_m^*(\nu)(1-\tilde{\delta}_m(n,\nu)\tilde{\delta}_m^*(n,\nu))\Bigg),\end{aligned} \tag{5.9}$$

$$\tilde{\delta}_m^*(n,\nu)\tilde{\delta}_m(n,\nu) = (\tilde{c}_{n-1,m}^*\tilde{c}_{n-1,m})\cdots(\tilde{c}_{\nu,m}^*\tilde{c}_{\nu,m}), \tag{5.10a}$$

$$1-\tilde{\delta}_m(n,\nu)\tilde{\delta}_m^*(n,\nu) = 1-(1-\tilde{c}_{n-1,m}^*\tilde{c}_{n-1,m})\cdots(1-\tilde{c}_{\nu,m}^*\tilde{c}_{\nu,m}). \tag{5.10b}$$

In order to calculate $[\ \widetilde{\Lambda}_0(n),\ \widetilde{\Lambda}_+(\nu)\ ]$, we must use the relation

$$\left[\ \sum_m \tilde{d}_m(\nu)\tilde{d}_m^*(\nu),\ \widetilde{\Lambda}_+(\nu)\ \right] = -\widetilde{\Lambda}_+(\nu). \tag{5.11}$$

For the derivation of the relation (5.11), we used the relations (3.18) and (3.19). Using relations (5.8b) and (5.11), we obtain the following:

$$\begin{aligned}[\ \widetilde{\Lambda}_0(n),\ \widetilde{\Lambda}_+(\nu)\ ] = &\frac{1}{2}\widetilde{\Lambda}_+(\nu)\\ &+\frac{1}{2}\Bigg(\sum_m[\ \tilde{d}_m^*(\nu)\tilde{d}_m(\nu),\ \widetilde{\Lambda}_+(\nu)\ ]\cdot\tilde{\delta}_m^*(n,\nu)\tilde{\delta}_m(n,\nu)\\ &\qquad+\sum_m[\ \tilde{d}_m(\nu)\tilde{d}_m^*(\nu),\ \widetilde{\Lambda}_+(\nu)\ ]\cdot(1-\tilde{\delta}_m(n,\nu)\tilde{\delta}_m^*(n,\nu))\Bigg).\end{aligned} \tag{5.12}$$

Successive use of the relation (5.12) gives us the relation

$$\begin{aligned}&\widetilde{\Lambda}_0(n)|N_{n-2}, N_{n-3}, \ldots, N_1, N_0\rangle = -\lambda(n)|N_{n-2}, N_{n-3}, \ldots, N_1, N_0\rangle,\\ &\lambda(n) = \Omega - \frac{1}{2}\left(\sum_{\nu=2}^{n-1}\frac{N_{\nu-1}-N_{\nu-2}}{\nu} + N_0\right)\\ &\qquad = \Omega - \frac{1}{2}\left(\sum_{\nu=2}^{n-1}\frac{N_{\nu-1}}{\nu(\nu+1)} + \left(\frac{N_{n-2}}{n} - \frac{N_0}{2}\right) + N_0\right).\end{aligned} \tag{5.13}$$

Here, we used the relation (5.8b) and

$$\begin{aligned}&\widetilde{\Lambda}_0(n)|N_0\rangle = \frac{1}{2}(N_0 - 2\Omega)|N_0\rangle,\\ &\tilde{\delta}_m^*(n,\nu)\tilde{\delta}_m(n,\nu)|N_0\rangle = 0, \qquad (1-\tilde{\delta}_m(n,\nu)\tilde{\delta}_m^*(n,\nu))|N_0\rangle = 0.\end{aligned} \tag{5.14}$$

Thus, we have learned that $|N_{n-2}, N_{n-3}, \ldots, N_1, N_0\rangle$ is the minimum weight state of $(\widetilde{\Lambda}_{\pm,0}(n))$. For the $[(N_{n-1} - N_{n-2})/n]$th-time operation of $\widetilde{\Lambda}_+(n)$ on this minimum weight state, we have the form (5.2):

$$|N_{n-1}, N_{n-2}, N_{n-3}, \ldots, N_1, N_0\rangle = \left(\widetilde{\Lambda}_+(n)\right)^{\frac{N_{n-1}-N_{n-2}}{n}} |N_{n-2}, N_{n-3}, \ldots, N_1, N_0\rangle$$
$$= \prod_{\nu=2}^{n} \left(\widetilde{\Lambda}_+(\nu)\right)^{\frac{N_{\nu-1}-N_{\nu-2}}{\nu}} |N_0\rangle, \tag{5.15}$$

$$\widetilde{\Lambda}_0(n)|N_{n-1}, N_{n-2}, N_{n-3}, \ldots, N_1, N_0\rangle$$
$$= \left(\frac{N_{n-1} - N_{n-2}}{n} - \lambda(n)\right) |N_{n-1}, N_{n-2}, N_{n-3}, \ldots, N_1, N_0\rangle. \tag{5.16}$$

Next, we will show that the state (5.15) is the minimum weight state of the $SU(n)$ Lipkin model. First, the following relations are derived from the relation (3.15):

$$[\, \widetilde{S}_p(n),\ \tilde{d}^*_{\lambda,m} \,] = \delta_{\lambda p}\tilde{d}^*_{0,m} \quad (p = 1, 2, \ldots, n-1), \tag{5.17a}$$

$$[\, \widetilde{S}^q_p(n),\ \tilde{d}^*_{\lambda,m} \,] = \delta_{\lambda p}\tilde{d}^*_{q,m} \quad (q < p = 2, 3, \ldots, n-1), \tag{5.17b}$$

$$[\, \widetilde{S}^p_p(n),\ \tilde{d}^*_{\lambda,m} \,] = (\delta_{\lambda p} - \delta_{\lambda 0})\tilde{d}^*_{\lambda,m} \quad (p = 1, 2, \ldots, n-1). \tag{5.17c}$$

With the use of the relation (5.17), we have

$$[\, \widetilde{S}_p(n),\ \widetilde{\Lambda}_+(\nu) \,] = 0, \qquad [\, \widetilde{S}^q_p(n),\ \widetilde{\Lambda}_+(\nu) \,] = 0, \tag{5.18}$$

$$[\, \widetilde{S}^p_p(n),\ \widetilde{\Lambda}_+(\nu) \,] = \begin{cases} 0 & (p \le \nu - 1), \\ -\widetilde{\Lambda}_+(\nu) & (p > \nu - 1). \end{cases} \tag{5.19}$$

Noting the relations $\widetilde{S}_p(n)|N_0\rangle = 0$, $\widetilde{S}^q_p(n)|N_0\rangle = 0$, and $\widetilde{S}^p_p(n)|N_0\rangle = -N_0|N_0\rangle$, we can show that the state (5.15) is the minimum weight state of the $SU(n)$ Lipkin model:

$$\widetilde{S}_p(n)|N_{n-1}, N_{n-2}, \ldots, N_1, N_0\rangle = 0, \tag{5.20a}$$

$$\widetilde{S}^q_p(n)|N_{n-1}, N_{n-2}, \ldots, N_1, N_0\rangle = 0, \tag{5.20b}$$

$$\widetilde{S}^p_p(n)|N_{n-1}, N_{n-2}, \ldots, N_1, N_0\rangle$$
$$= -\left(\sum_{\nu=1}^{p} \frac{N_{\nu-1} - N_{\nu-2}}{\nu}\right) |N_{n-1}, N_{n-2}, \ldots, N_1, N_0\rangle \quad (N_{-1} = 0). \tag{5.21}$$

Thus, we could find the minimum weight state for the general case.

In the relations (4.11), (4.12), (4.25), and (4.26), we showed the inequalities that the fermion numbers $N_{\nu-1}$ and $\gamma_{\nu-1}(p)$ in the cases of the $SU(2)$ and the $SU(3)$ Lipkin model should satisfy. As the final remark of this section, we will give the inequalities for the general case. First, the relation between N_{n-1} and $\gamma_{n-1}(p)$ for the $SU(n)$ Lipkin model must be discussed. The minimum weight state $|\min(n)\rangle = |N_{n-1}, N_{n-2}, \ldots, N_1, N_0\rangle$ shown in the relation (5.16) gives us the following relation:

$$\gamma_{n-1}(0) = \frac{N_{n-1} - N_{n-2}}{n} + \frac{N_{n-2} - N_{n-3}}{n-1} + \cdots + \frac{N_2 - N_1}{3} + \frac{N_1 - N_0}{2} + N_0, \tag{5.22a}$$

$$\gamma_{n-1}(1) = \frac{N_{n-1} - N_{n-2}}{n} + \frac{N_{n-2} - N_{n-3}}{n-1} + \cdots + \frac{N_2 - N_1}{3} + \frac{N_1 - N_0}{2},$$

$$\vdots$$
$$\gamma_{n-1}(n-2) = \frac{N_{n-1}-N_{n-2}}{n} + \frac{N_{n-2}-N_{n-3}}{n-1},$$
$$\gamma_{n-1}(n-1) = \frac{N_{n-1}-N_{n-2}}{n}. \tag{5.22b}$$

The relation (5.22) is written compactly as

$$\gamma_{n-1}(p) = \begin{cases} \displaystyle\sum_{\nu=2}^{n} \frac{N_{\nu-1}-N_{\nu-2}}{\nu} + N_0 \quad (p=0), & (5.23a) \\ \displaystyle\sum_{\nu=p+1}^{n} \frac{N_{\nu-1}-N_{\nu-2}}{\nu} \quad (p=1,2,\ldots,n-1). & (5.23b) \end{cases}$$

The relation (5.23) is inversely expressed as

$$N_\nu = \begin{cases} \displaystyle\sum_{p=0}^{\nu} \gamma_{n-1}(p) - (\nu+1)\gamma_{n-1}(\nu+1) \quad (\nu=0,1,\ldots,n-2), & (5.24a) \\ \displaystyle\sum_{p=0}^{n-1} \gamma_{n-1}(p) \quad (\nu=n-1). & (5.24b) \end{cases}$$

The form (5.24b) is nothing but the relation (2.20). We can rewrite (5.22) to the following:

$$\gamma_{n-1}(0) - \gamma_{n-1}(1) = N_0 \quad (p=0), \tag{5.25a}$$
$$\gamma_{n-1}(p) - \gamma_{n-1}(p+1) = \frac{N_p - N_{p-1}}{p+1} \quad (p=1,2,\ldots,n-2), \tag{5.25b}$$
$$\gamma_{n-1}(n-1) = \frac{N_{n-1}-N_{n-2}}{n} \quad (p=n-1). \tag{5.25c}$$

The right-hand side of the relation (5.25) should be zero or positive, and then we have

$$0 \le N_0 \le N_1 \le \cdots \le N_{n-2} \le N_{n-1}, \tag{5.26}$$
$$0 \le \gamma_{n-1}(n-1) \le \gamma_{n-1}(n-2) \le \cdots \le \gamma_{n-1}(1) \le \gamma_{n-1}(0). \tag{5.27}$$

At the present, the upper limit cannot be determined.

To determine the upper limit, we note that $(\widetilde{\Lambda}_{\pm,0}(n))$ obeys the $SU(2)$ algebra, and the relations (5.13) and (5.16) give us the following inequalities:

$$\lambda(n) \ge 0,$$
$$\text{i.e.,}\ \Omega - \frac{1}{2}\left(\sum_{\nu=2}^{n-1} \frac{N_{\nu-1}}{\nu(\nu+1)} + \left(\frac{N_{n-2}}{n} - \frac{N_0}{2}\right) + N_0\right) \ge 0, \tag{5.28}$$

$$-\lambda(n) \le \frac{N_{n-1}-N_{n-2}}{n} - \lambda(n) \le \lambda(n), \tag{5.29a}$$
$$\text{i.e.,}\ N_{n-2} \le N_{n-1} \le n\left(2\Omega - \sum_{\nu=1}^{n-1} \frac{N_{\nu-1}}{\nu(\nu+1)}\right). \tag{5.29b}$$

The relation (5.28) combined with the relation (5.26) lead us to

$$N_0 \leq 2\Omega \quad (n=2), \tag{5.30a}$$

$$N_{n-2} \leq (n-1)\left(2\Omega - \sum_{\nu=1}^{n-2}\frac{N_{\nu-1}}{\nu(\nu+1)}\right) \quad (n=3,4,\ldots), \tag{5.30b}$$

$$0 \leq N_0 \leq N_1 \leq \cdots \leq N_{n-1} \leq n\left(2\Omega - \sum_{\nu=1}^{n-1}\frac{N_{\nu-1}}{\nu(\nu+1)}\right) \quad (n=2,3,\ldots). \tag{5.31}$$

The relations (5.29) and (5.30) for the cases $n=2$ and 3 reduce to the relations (4.11) and (4.25), respectively. The inequality (5.29a) leads us to the following:

$$\gamma_{n-1}(0) \leq 2\Omega. \tag{5.32}$$

For the derivation, we used the relation (5.23). Then, we have

$$0 \leq \gamma_{n-1}(n-1) \leq \gamma_{n-1}(n-2) \leq \cdots \leq \gamma_{n-1}(1) \leq \gamma_{n-1}(0) \leq 2\Omega. \tag{5.33}$$

Thus, we could present the minimum weight state of the general case. It should be noted that all the relations given in this section are available for $n \geq 3$.

6. Discussions

So far, we have developed a possible idea for how to give concrete expressions of the minimum weight states for the $SU(n)$ Lipkin model for arbitrary fermion number. In this section, we will treat some simple examples of the minimum weight states from a viewpoint slightly different from that in the last section. This argument is also in preparation for the next paper (II). Our discussion starts by mentioning that the $SU(n)$ Lipkin model contains $SU(2)$ subalgebras, the number depending on the number n. In this section, we will discuss the cases $n=2$, 3, 4, and 5. The case $n=2$ is the $SU(2)$ algebra itself and the case $n=3$ has one $SU(2)$ subalgebra. On the other hand, the cases $n=4$ and 5 contain two $SU(2)$ algebras. One by one, we will show this.

In the case $n=2$, $\widetilde{S}^1 (=\widetilde{S}_+)$, $\widetilde{S}_1 (=\widetilde{S}_-)$, and $\widetilde{S}_1^1/2 (=\widetilde{S}_0)$ form the $SU(2)$ algebra and $\widetilde{\Gamma}_{SU(2)}$ is given as

$$\widetilde{\Gamma}_{SU(2)} = \widetilde{S}_+\widetilde{S}_- + \widetilde{S}_0\left(\widetilde{S}_0 - 1\right). \tag{6.1}$$

The above is nothing but the original Lipkin model. The minimum weight state $|\mathrm{min}(2)\rangle$ is specified by two quantum numbers N and s, the eigenvalues of $\widetilde{N}$ and $-\widetilde{S}_0$: $|\mathrm{min}(2)\rangle = |N;s\rangle$. Of course, these two are related to the algebra. Then, for the orthogonal set, we have

$$|N;ss_0\rangle = \sqrt{\frac{(s-s_0)!}{(2s)!(s+s_0)!}}\left(\widetilde{S}_+\right)^{s+s_0}|N;s\rangle. \tag{6.2}$$

In this case, we obtain the relation

$$\gamma_1(0) = \frac{N}{2} + s\ (\geq 0), \qquad \gamma_1(1) = \frac{N}{2} - s\ (\geq 0). \tag{6.3}$$

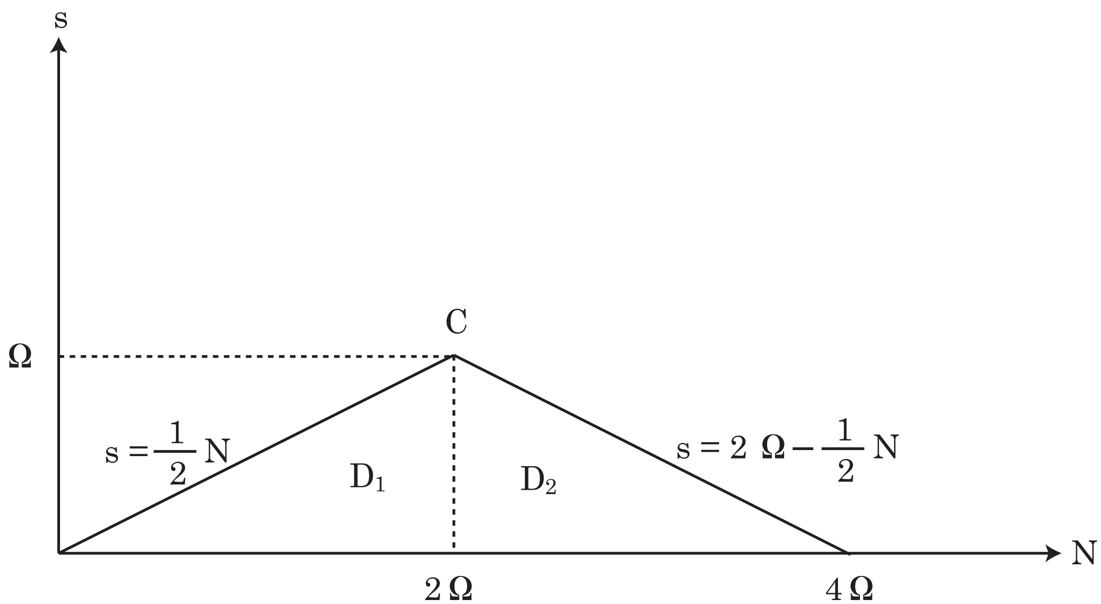


Fig. 2. The relation between s and N is shown in the inequality (6.4) in the case $n = 2$, namely, in the case of the $SU(2)$ Lipkin model.

Then, using the inequality (4.12), we can show that the relation (6.3) holds in the following domains:

$$(\mathrm{D}_1)\ \ 0 \le N \le 2\Omega, \qquad 0 \le s \le \frac{N}{2}, \tag{6.4a}$$

$$(\mathrm{D}_2)\ \ 2\Omega \le N \le 4\Omega, \qquad 0 \le s \le 2\Omega - \frac{N}{2}. \tag{6.4b}$$

The above domains are illustrated in Fig. 2. A "closed-shell" system appears in the case ($N = 2\Omega$, $s = \Omega$), where, in the absence of interactions, the level $p = 0$ is occupied fully by the fermions and the level $p = 1$ is vacant. Point C in Fig. 2 corresponds to the "closed-shell" system. But, s can decrease from $s = \Omega$ to $s = 0$, where the levels $p = 0$ and $p = 1$ are occupied in equal fermion number Ω.

Next, we treat the case $n = 3$. The operators $\widetilde{S}_1^2(= \widetilde{S}_+)$, $\widetilde{S}_2^1(= \widetilde{S}_-)$, and $(\widetilde{S}_2^2 - \widetilde{S}_1^1)/2(= \widetilde{S}_0)$ form the $SU(2)$ subalgebra and, further, we have the scalar $\widetilde{R}_0$ with respect to $(\widetilde{S}_{\pm,0})$ in the form

$$\widetilde{R}_0 = \frac{1}{2}\left(\widetilde{S}_2^2 + \widetilde{S}_1^1\right) \qquad \left(\left[\,\widetilde{S}_{\pm,0},\ \widetilde{R}_0\,\right] = 0\,\right). \tag{6.5}$$

The Casimir operator $\widetilde{\Gamma}_{SU(3)}$ is expressed as

$$\widetilde{\Gamma}_{SU(3)} = \left(\widetilde{S}^2\widetilde{S}_2 + \widetilde{S}^1\widetilde{S}_1\right) + \left(\widetilde{S}_+\widetilde{S}_- + \widetilde{S}_0\left(\widetilde{S}_0 - 1\right)\right) + \frac{1}{3}\widetilde{R}_0\left(\widetilde{R}_0 - 3\right). \tag{6.6}$$

In addition to N, $|\min(3)\rangle$ can be specified by the eigenvalues of $\widetilde{S}_0$ and $\widetilde{R}_0$, $-\sigma$ and $-\rho$, respectively: $|\min(3)\rangle = |N; \rho, \sigma\rangle$. Then, we have

$$|N; \rho, \sigma\sigma_0\rangle = \sqrt{\frac{(\sigma - \sigma_0)!}{(2\sigma)!(\sigma + \sigma_0)!}}\left(\widetilde{S}_+\right)^{\sigma+\sigma_0} |N; \rho, \sigma\rangle. \tag{6.7}$$

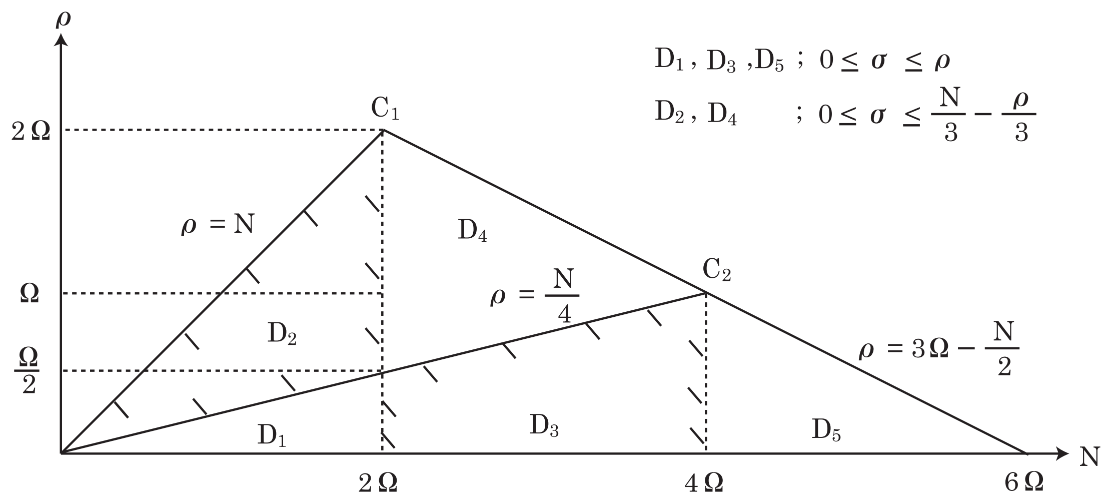


Fig. 3. The domains depicted in (6.9) are illustrated in the case $n = 3$, namely, in the case of the $SU(3)$ Lipkin model.

Therefore, for constructing the orthogonal sets, we, further, must take account of $(\widetilde{S}^2, \widetilde{S}^1)$; this will be discussed in (II). For now, we have the relation

$$\gamma_2(0) = \frac{N}{3} + \frac{2\rho}{3}, \qquad \gamma_2(1) = \frac{N}{3} - \frac{\rho}{3} + \sigma, \qquad \gamma_2(2) = \frac{N}{3} - \frac{\rho}{3} - \sigma. \tag{6.8}$$

The inequality (4.23) leads us to the following domains for the relation (6.8):

(i) $0 \le N \le 2\Omega$

$$(\mathrm{D}_1)\ \ 0 \le \rho \le \frac{N}{4},\ \ 0 \le \sigma \le \rho, \quad (\mathrm{D}_2)\ \ \frac{N}{4} \le \rho \le N,\ \ 0 \le \sigma \le \frac{N}{3} - \frac{\rho}{3}, \tag{6.9a}$$

(ii) $2\Omega \le N \le 4\Omega$

$$(\mathrm{D}_3)\ \ 0 \le \rho \le \frac{N}{4},\ \ 0 \le \sigma \le \rho, \quad (\mathrm{D}_4)\ \ \frac{N}{4} \le \rho \le 3\Omega - \frac{N}{2},\ \ 0 \le \sigma \le \frac{N}{3} - \frac{\rho}{3}, \tag{6.9b}$$

(iii) $4\Omega \le N \le 6\Omega$

$$(\mathrm{D}_5)\ \ 0 \le \rho \le 3\Omega - \frac{N}{2},\ \ 0 \le \sigma \le \rho. \tag{6.9c}$$

The above domains are illustrated in Fig. 3. The present case contains two “closed-shell” systems. The first appears at the point C_1 in Fig. 3 ($N = 2\Omega,\ \rho = 2\Omega,\ \sigma = 0$). Only level $p = 0$ is occupied. The second appears at the point C_2 in Fig. 3 ($N = 4\Omega,\ \rho = \Omega,\ \sigma = \Omega$). In this case, the levels $p = 0$ and 1 are fully occupied. However, by changing the values of ρ and σ, we can produce various fermion number distributions.

Third, we have the case $n = 4$, where there exist two $SU(2)$ subalgebras: $\widetilde{S}_2^3 (= \widetilde{S}_+(1))$, $\widetilde{S}_3^2 (= \widetilde{S}_-(1))$, $(\widetilde{S}_3^3 - \widetilde{S}_2^2)/2 (= \widetilde{S}_0(1))$ and $\widetilde{S}^1 (= \widetilde{S}_+(2))$, $\widetilde{S}_1 (= \widetilde{S}_-(2))$, $\widetilde{S}_1^1/2 (= \widetilde{S}_0(2))$. Further, we denote the addition of the above two as

$$\widetilde{S}_{\pm,0} = \widetilde{S}_{\pm,0}(1) + \widetilde{S}_{\pm,0}(2). \tag{6.10}$$

This case gives us one scalar with respect to $(\widetilde{S}_{\pm,0})$:

$$\widetilde{R}_0 = \frac{1}{2}\left(\widetilde{S}_3^3 + \widetilde{S}_2^2 - \widetilde{S}_1^1\right). \tag{6.11}$$

The Casimir operator $\widetilde{\Gamma}_{SU(4)}$ is written as

$$\widetilde{\Gamma}_{SU(4)} = \left(\widetilde{S}^3\widetilde{S}_3 + \widetilde{S}^2\widetilde{S}_2 + \widetilde{S}_1^3\widetilde{S}_3^1 + \widetilde{S}_1^2\widetilde{S}_2^1\right) + \sum_{i=1,2}\left(\widetilde{S}_+(i)\widetilde{S}_-(i) + \widetilde{S}_0(i)\left(\widetilde{S}_0(i)-1\right)\right) + \frac{1}{2}\widetilde{R}_0\left(\widetilde{R}_0 - 4\right). \tag{6.12}$$

The minimum weight state $|\min(4)\rangle$ can be expressed as $|N;\rho,\sigma^1,\sigma^2\rangle$. Here, of course, ρ, σ^1, and σ^2 denote the eigenvalues of $-\widetilde{R}_0$, $-\widetilde{S}_0(1)$, and $-\widetilde{S}_0(2)$, respectively. Then, we have the following state:

$$|N;\rho,\sigma^1,\sigma^2,\sigma\sigma_0\rangle = \sum_{\sigma_0^1,\sigma_0^2}\langle\sigma^1\sigma_0^1,\sigma^2\sigma_0^2|\sigma\sigma_0\rangle\sqrt{\frac{(\sigma^1-\sigma_0^1)!}{(2\sigma^1)!(\sigma^1+\sigma_0^1)!}}\sqrt{\frac{(\sigma^2-\sigma_0^2)!}{(2\sigma^2)!(\sigma^2+\sigma_0^2)!}} \times \left(\widetilde{S}_+(1)\right)^{\sigma^1+\sigma_0^1}\left(\widetilde{S}_+(2)\right)^{\sigma^2+\sigma_0^2}|N;\rho,\sigma^1,\sigma^2\rangle. \tag{6.13}$$

Then, the role of $\widetilde{S}^3$, $\widetilde{S}^2$, $\widetilde{S}_1^3$, and $\widetilde{S}_1^2$ becomes interesting for constructing the orthogonal sets. In the present case, we can derive the relation

$$\gamma_4(0) = \frac{N}{4} + \frac{\rho}{2} + \sigma^2, \qquad \gamma_4(1) = \frac{N}{4} + \frac{\rho}{2} - \sigma^2,$$
$$\gamma_4(2) = \frac{N}{4} - \frac{\rho}{2} + \sigma^1, \qquad \gamma_4(3) = \frac{N}{4} - \frac{\rho}{2} - \sigma^1. \tag{6.14}$$

The inequality (5.33) gives the following 12 domains:

(i) $0 \le N \le 2\Omega$

$$(\mathrm{D}_1)\ 0 \le \rho \le \frac{N}{6}, \qquad (\mathrm{D}_2)\ \frac{N}{6} \le \rho \le \frac{N}{2}, \tag{6.15a}$$

(ii) $2\Omega \le N \le 4\Omega$

$$(\mathrm{D}_3)\ 0 \le \rho \le \frac{N}{6}, \qquad (\mathrm{D}_4)\ \frac{N}{6} \le \rho \le \frac{4\Omega}{3} - \frac{N}{6},$$
$$(\mathrm{D}_5)\ \frac{4\Omega}{3} - \frac{N}{6} \le \rho \le \Omega, \qquad (\mathrm{D}_6)\ \Omega \le \rho \le \frac{N}{2}, \tag{6.15b}$$

(iii) $4\Omega \le N \le 6\Omega$

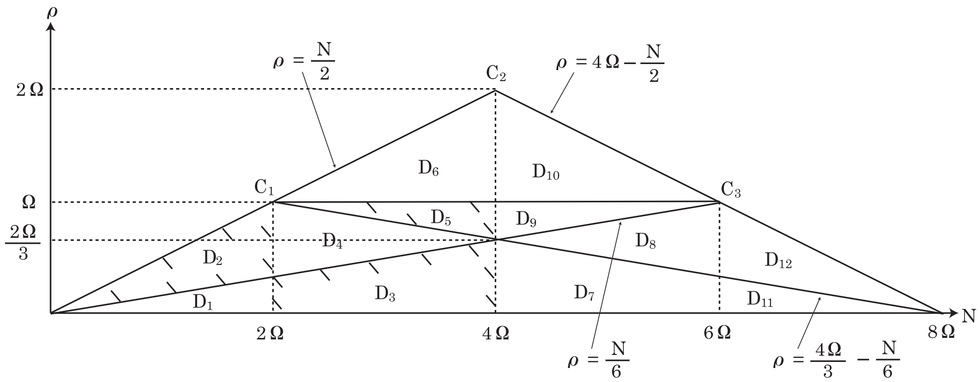
$$(\mathrm{D}_7)\ 0 \le \rho \le \frac{4\Omega}{3} - \frac{N}{6}, \qquad (\mathrm{D}_8)\ \frac{4\Omega}{3} - \frac{N}{6} \le \rho \le \frac{N}{6},$$
$$(\mathrm{D}_9)\ \frac{N}{6} \le \rho \le \Omega, \qquad (\mathrm{D}_{10})\ \Omega \le \rho \le 4\Omega - \frac{N}{2}, \tag{6.15c}$$

(iv) $6\Omega \le N \le 8\Omega$

$$(\mathrm{D}_{11})\ 0 \le \rho \le 4\Omega - \frac{N}{2}, \qquad (\mathrm{D}_{12})\ \frac{4\Omega}{3} - \frac{N}{6} \le \rho \le 4\Omega - \frac{N}{2}. \tag{6.15d}$$

These are illustrated in the complicated Fig. 4. Three "closed-shell" systems appear in this case: C_1 ($N = 2\Omega$, $\rho = \Omega$, $\sigma^1 = 0$, $\sigma^2 = 0$), C_2 ($N = 4\Omega$, $\rho = 2\Omega$, $\sigma^1 = 0$, $\sigma^2 = 0$), and C_3 ($N = 6\Omega$, $\rho = \Omega$, $\sigma^1 = \Omega$, $\sigma^2 = 0$). By changing the values of ρ, σ^1, and σ^2, we can produce various fermion number distributions.

Finally, we will treat the case $n = 5$. In this case, we also have two $SU(2)$ subalgebras: $\widetilde{S}_3^4(=\widetilde{S}_+(1))$, $\widetilde{S}_4^3(=\widetilde{S}_-(1))$, $(\widetilde{S}_4^4 - \widetilde{S}_3^3)/2(=\widetilde{S}_0(1))$ and $\widetilde{S}_1^2(=\widetilde{S}_+(2))$, $\widetilde{S}_2^1(=\widetilde{S}_-(2))$, $(\widetilde{S}_2^2 - \widetilde{S}_1^1)/2(=\widetilde{S}_0(2))$. For



D_1, D_3, D_7, D_{11} ; $0 \leq \sigma^1 \leq \rho$, $0 \leq \sigma^2 \leq \rho - \sigma^1$

D_2, D_4 ; $0 \leq \sigma^1 \leq \frac{N}{4} - \frac{\rho}{2}$, $0 \leq \sigma^2 \leq \rho - \sigma^1$

D_8, D_{12} ; $\frac{N}{4} - 2\Omega + \frac{3\rho}{2} \leq \sigma^1 \leq \rho$, $0 \leq \sigma^2 \leq \rho - \sigma^1$

D_5, D_9 ; $0 \leq \sigma^1 \leq \frac{N}{4} - 2\Omega + \frac{3\rho}{2}$, $0 \leq \sigma^2 \leq 2\Omega - \frac{N}{4} - \frac{\rho}{2}$
or $\frac{N}{4} - 2\Omega + \frac{3\rho}{2} \leq \sigma^1 \leq \frac{N}{4} - \frac{\rho}{2}$, $0 \leq \sigma^2 \leq \rho - \sigma^1$

D_6, D_{10} ; $0 \leq \sigma^1 \leq \frac{N}{4} - \frac{\rho}{2}$, $0 \leq \sigma^2 \leq 2\Omega - \frac{N}{4} - \frac{\rho}{2}$

Fig. 4. The domains depicted in (6.15) are illustrated in the case $n = 4$, namely, in the case of the $SU(4)$ Lipkin model.

these two, we also use $\widetilde{S}_{\pm,0}$ given in the relation (6.10). However, the present case contains two scalars with respect to $(\widetilde{S}_{\pm,0})$:

$$\widetilde{R}_0(1) = \frac{1}{2}\left(\widetilde{S}_4^4 + \widetilde{S}_3^3 - \widetilde{S}_2^2 - \widetilde{S}_1^1\right), \qquad \widetilde{R}_0(2) = \frac{1}{2}\left(\widetilde{S}_4^4 + \widetilde{S}_3^3 + \widetilde{S}_2^2 + \widetilde{S}_1^1\right). \tag{6.16}$$

The Casimir operator $\widetilde{\Gamma}_{SU(5)}$ can be expressed as

$$\begin{aligned}\widetilde{\Gamma}_{SU(5)} &= \left(\widetilde{S}^4\widetilde{S}_4 + \widetilde{S}^3\widetilde{S}_3 + \widetilde{S}^2\widetilde{S}_2 + \widetilde{S}^1\widetilde{S}_1 + \widetilde{S}_1^4\widetilde{S}_4^1 + \widetilde{S}_1^3\widetilde{S}_3^1 + \widetilde{S}_2^4\widetilde{S}_4^2 + \widetilde{S}_2^3\widetilde{S}_3^2\right) \\ &\quad + \sum_{i=1,2}\left(\widetilde{S}_+(i)\widetilde{S}_-(i) + \widetilde{S}_0(i)\left(\widetilde{S}_0(i) - 1\right)\right) \\ &\quad + \frac{1}{2}\widetilde{R}_0(1)\left(\widetilde{R}_0(1) - 4\right) + \frac{1}{10}\widetilde{R}_0(2)\left(\widetilde{R}_0(2) - 10\right).\end{aligned} \tag{6.17}$$

The minimum weight state is specified by ρ^1, ρ^2, σ^1, and σ^2, which are the eigenvalues of $-\widetilde{R}_0(1)$, $-\widetilde{R}_0(2)$, $-\widetilde{S}_0(1)$, and $-\widetilde{S}_0(2)$, respectively: $|\min(5)\rangle = |N; \rho^1, \rho^2, \sigma^1, \sigma^2\rangle$. Then, we have

$$\begin{aligned}|N; \rho^1, \rho^2, \sigma^1, \sigma^2, \sigma\sigma_0\rangle &= \sum_{\sigma_0^1, \sigma_0^2} \langle \sigma^1\sigma_0^1\sigma^2\sigma_0^2|\sigma\sigma_0\rangle \sqrt{\frac{(\sigma^1 - \sigma_0^1)!}{(2\sigma^1)!(\sigma^1 + \sigma_0^1)!}}\sqrt{\frac{(\sigma^2 - \sigma_0^2)!}{(2\sigma^2)!(\sigma^2 + \sigma_0^2)!}} \\ &\quad \times \left(\widetilde{S}_+(1)\right)^{\sigma^1 + \sigma_0^1}\left(\widetilde{S}_+(2)\right)^{\sigma^2 + \sigma_0^2}|N; \rho^1, \rho^2, \sigma^1, \sigma^2\rangle.\end{aligned} \tag{6.18}$$

Of course, the roles of $\widetilde{S}^4$, $\widetilde{S}^3$, $\widetilde{S}^2$, $\widetilde{S}^1$, $\widetilde{S}_1^4$, $\widetilde{S}_1^3$, $\widetilde{S}_2^4$, and $\widetilde{S}_2^3$ must be investigated. In the case $n = 5$, we have the following relation:

$$\begin{aligned}&\gamma_5(0) = \frac{N}{5} + \frac{2}{5}\rho^2, \\ &\gamma_5(1) = \frac{N}{5} + \frac{1}{2}\rho^1 - \frac{1}{10}\rho^2 + \sigma^2, \qquad \gamma_5(2) = \frac{N}{5} + \frac{1}{2}\rho^1 - \frac{1}{10}\rho^2 - \sigma^2, \\ &\gamma_5(3) = \frac{N}{5} - \frac{1}{2}\rho^1 - \frac{1}{10}\rho^2 + \sigma^1, \qquad \gamma_5(4) = \frac{N}{5} - \frac{1}{2}\rho^1 - \frac{1}{10}\rho^2 - \sigma^1.\end{aligned} \tag{6.19}$$

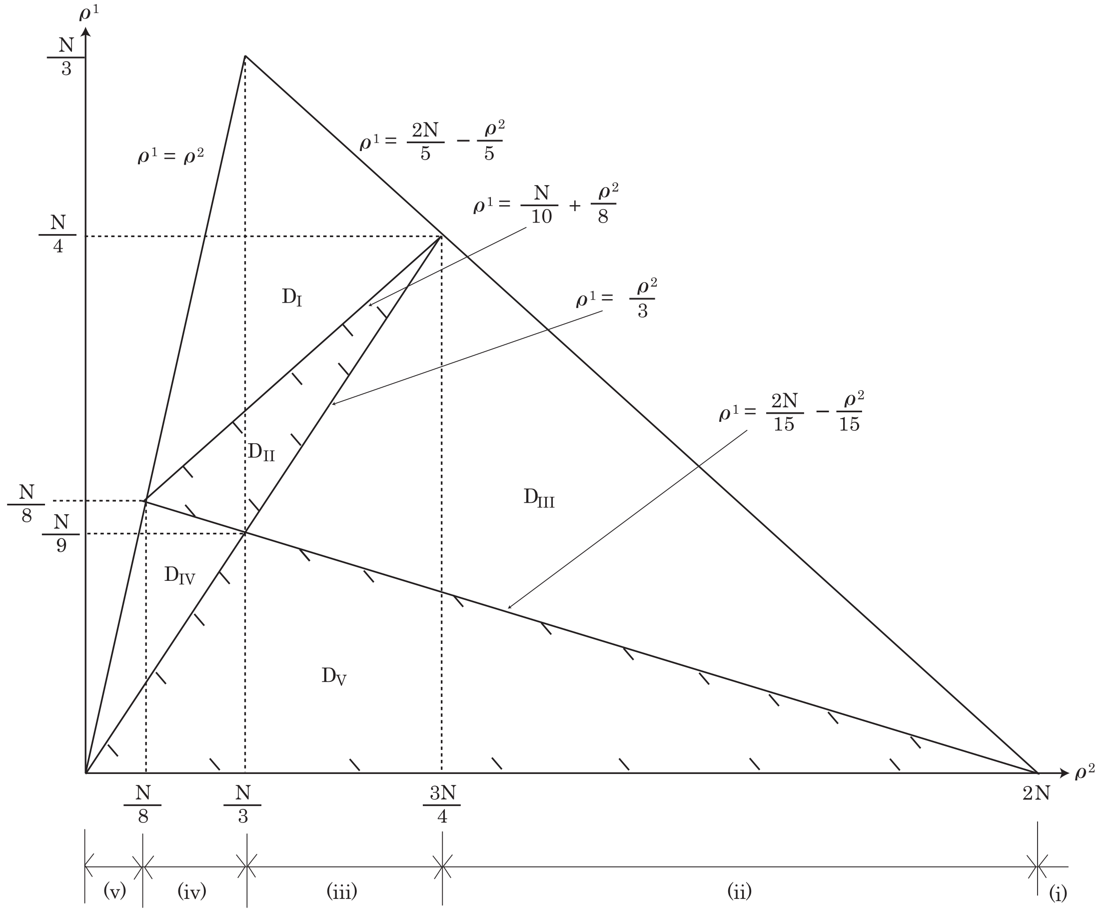


Fig. 5. The domains depicted in (6.20) are illustrated in the case $n = 5$, namely, in the case of the $SU(5)$ Lipkin model.

In this case, we also use the inequality (5.33). But, different from the case $n = 4$, we cannot give the relations between N and ρ^1 also between N and ρ^2, respectively. We give the relation between ρ^1 and ρ^2 by regarding N as a parameter. Inequality (5.33), except for $\gamma_5(0) \leq 2\Omega$, leads us to the following:

$$(\mathrm{D_I}) \quad \rho^1 \geq \frac{2N}{15} - \frac{\rho^2}{15}, \qquad \rho^1 \geq \frac{\rho^2}{3}, \qquad \rho^1 \geq \frac{N}{10} + \frac{\rho^2}{5}, \tag{6.20a}$$

$$(\mathrm{D_{II}}) \quad \rho^1 \geq \frac{2N}{15} - \frac{\rho^2}{15}, \qquad \rho^1 \geq \frac{\rho^2}{3}, \qquad \rho^1 \leq \frac{N}{10} + \frac{\rho^2}{5}, \tag{6.20b}$$

$$(\mathrm{D_{III}}) \quad \rho^1 \geq \frac{2N}{15} - \frac{\rho^2}{15}, \qquad \rho^1 \leq \frac{\rho^2}{3}, \tag{6.20c}$$

$$(\mathrm{D_{IV}}) \quad \rho^1 \leq \frac{2N}{15} - \frac{\rho^2}{15}, \qquad \rho^1 \geq \frac{\rho^2}{3}, \tag{6.20d}$$

$$(\mathrm{D_V}) \quad \rho^1 \leq \frac{2N}{15} - \frac{\rho^2}{15}, \qquad \rho^1 \leq \frac{\rho^2}{3}. \tag{6.20e}$$

The relation (6.20) is illustrated in Fig. 5. In each domain, σ^1 and σ^2 obey the inequalities

$$(\mathrm{D_I}) \quad 0 \leq \sigma^1 \leq \frac{N}{5} - \frac{\rho^2}{10} - \frac{\rho^1}{2}, \qquad 0 \leq \sigma^2 \leq \frac{\rho^2}{2} - \frac{\rho^1}{2}, \tag{6.21a}$$

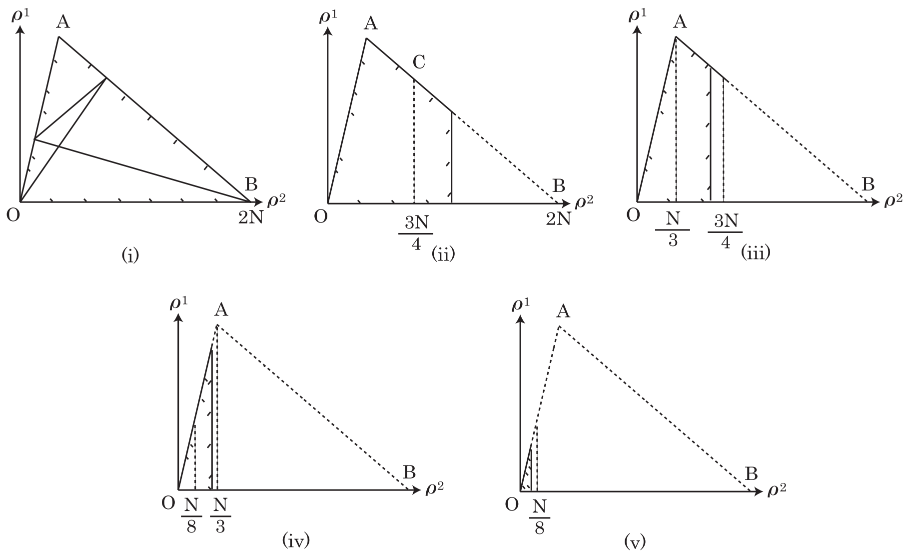


Fig. 6. The relation (6.21) is applied to the domains surrounded by short oblique lines in the case $n = 5$, namely, in the case of the $SU(5)$ Lipkin model.

$$
\begin{aligned}
&(\mathrm{D_{II}}) \quad 0 \le \sigma^1 \le \frac{3\rho^1}{2} - \frac{\rho^2}{2}, \qquad 0 \le \sigma^2 \le \frac{\rho^2}{2} - \frac{\rho^1}{2}, \\
&\text{or} \quad \frac{3\rho^1}{2} - \frac{\rho^2}{2} \le \sigma^1 \le \frac{N}{5} - \frac{\rho^2}{10} - \frac{\rho^1}{2}, \qquad 0 \le \sigma^2 \le \rho^1 - \sigma^1, \qquad (6.21\mathrm{b}) \\
&(\mathrm{D_{III}}) \quad 0 \le \sigma^1 \le \frac{N}{5} - \frac{\rho^2}{10} - \frac{\rho^1}{2}, \qquad 0 \le \sigma^2 \le \rho^1 - \sigma^1, \qquad (6.21\mathrm{c}) \\
&(\mathrm{D_{IV}}) \quad 0 \le \sigma^1 \le \frac{3\rho^1}{2} - \frac{\rho^2}{2}, \qquad 0 \le \sigma^2 \le \frac{\rho^2}{2} - \frac{\rho^1}{2}, \\
&\text{or} \quad \frac{3\rho^1}{2} - \frac{\rho^2}{2} \le \sigma^1 \le \rho^1, \qquad 0 \le \sigma^2 \le \rho^1 - \sigma^1, \qquad (6.21\mathrm{d}) \\
&(\mathrm{D_{V}}) \quad 0 \le \sigma^1 \le \rho^1, \qquad 0 \le \sigma^2 \le \rho^1 - \sigma^1. \qquad (6.21\mathrm{e})
\end{aligned}
$$

The inequality $\gamma_5(0) \le 2\Omega$ gives us the relation

$$\rho^2 \le 5\Omega - \frac{N}{2} \ (= \rho). \qquad (6.22)$$

The relation (6.22) does not depend on ρ^1, σ^1, or σ^2.

Combining ρ defined in the relation (6.22) with the regions (i)–(v) in the ρ^2-axis of Fig. 5, we have

$$
\begin{aligned}
&(\mathrm{v}) \ \ 0 \le \rho \le \frac{N}{8}, \quad (\mathrm{iv}) \ \ \frac{N}{8} \le \rho \le \frac{N}{3}, \quad (\mathrm{iii}) \ \ \frac{N}{3} \le \rho \le \frac{3N}{4}, \\
&(\mathrm{ii}) \ \ \frac{3N}{4} \le \rho \le 2N, \quad (\mathrm{i}) \ \ 2N \le \rho. \qquad (6.23\mathrm{a})
\end{aligned}
$$

The relation (6.23a) is reduced to

$$
\begin{aligned}
&(\mathrm{i}) \ \ 0 \le N \le 2\Omega, \quad (\mathrm{ii}) \ \ 2\Omega \le N \le 4\Omega, \quad (\mathrm{iii}) \ \ 4\Omega \le N \le 6\Omega, \\
&(\mathrm{iv}) \ \ 6\Omega \le N \le 8\Omega, \quad (\mathrm{v}) \ \ 8\Omega \le N \le 10\Omega. \qquad (6.23\mathrm{b})
\end{aligned}
$$

The relation (6.23b) is arranged in the inverted order. It does not necessarily follow that each region covers the whole domains shown in Fig. 5 (△OAB in Fig. 6). We show this feature in Fig. 6.

The relation (6.21) should be applied to the domains surrounded by short oblique lines in Fig. 6. Four "closed-shell" systems appear in the present case: C_1 ($N = 2\Omega,\ \rho^2 = 4\Omega,\ \rho^1 = 0,\ \sigma^2 = \sigma^1 = 0$), C_2 ($N = 4\Omega,\ \rho^2 = 3\Omega,\ \rho^1 = \Omega,\ \sigma^2 = \Omega,\ \sigma^1 = 0$), C_3 ($N = 6\Omega,\ \rho^2 = 2\Omega,\ \rho^1 = 0,\ \sigma^2 = \sigma^1 = 0$), and C_4 ($N = 8\Omega,\ \rho^2 = \Omega,\ \rho^1 = \Omega,\ \sigma^2 = 0,\ \sigma^1 = \Omega$). By changing the values of ρ^2, ρ^1, σ^2, and σ^1, we can produce various fermion number distributions.

In this section, we have presented the structure of the minimum weight states for the cases $n = 2$–5 in a form slightly different from that given in Sect. 5. The basic idea comes from the introduction of the $SU(2)$ subalgebras and the scalar operators defined in the relations (6.5), (6.18), and (6.16). In (II), we will discuss the cases with arbitrary values of n. Of course, the scalar operators are generalized.

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Appendix. A possible example of the state $|N_0\rangle$ introduced in the relation (4.1)

In this appendix, the state $|N_0\rangle$ is presented through the eigenvalue problem of $(\widetilde{\Lambda}_{\pm,0}(1))$ defined in the relation (3.12). The level $p = 0$ consists of 2Ω single-particle states $m = -j, -j+1, \dots, j-1, j$ ($2\Omega = 2j+1$). These states can be divided into two groups. One consists of $(m_1, m_2, \dots, m_\Omega)$, and the other $(\bar{m}_1, \bar{m}_2, \dots, \bar{m}_\Omega)$. We regard the state $\bar{m}_i$ as the partner of m_i ($i = 1, 2, \dots, \Omega$). The choice is arbitrary and, for example, all of the m_i and $\bar{m}_i$ are positive and negative, respectively. Under the above classification, we define the state

$$|m_\nu, m_{\nu-1}, \dots, m_2, m_1\rangle\rangle = \tilde{c}^*_{m_\nu}\tilde{c}^*_{m_{\nu-1}}\cdots\tilde{c}^*_{m_2}\tilde{c}^*_{m_1}|0\rangle. \tag{A.1}$$

Here, the index $p = 0$ was omitted and we fix the ordering of m_i appropriately, for example, $m_\nu > m_{\nu-1} > \cdots > m_2 > m_1$. It may be self-evident that the state (A.1) is not the minimum weight state of $(\widetilde{\Lambda}_{\pm,0}(1))$. Then, by replacing $\tilde{c}^*_m$ with $\widetilde{D}^*_m$, we introduce the following state:

$$|m_\nu, m_{\nu-1}, \dots, m_2, m_1\rangle = \widetilde{D}^*_{m_\nu}\widetilde{D}^*_{m_{\nu-1}}\cdots\widetilde{D}^*_{m_2}\widetilde{D}^*_{m_1}|0\rangle, \tag{A.2}$$

$$\widetilde{D}^*_m = \frac{1}{\sqrt{2}}(\tilde{d}^*_m - \tilde{d}^*_{\bar{m}}) = \frac{1}{\sqrt{2}}(e_m\tilde{c}^*_m - e_{\bar{m}}\tilde{c}^*_{\bar{m}}). \tag{A.3}$$

The operator $\widetilde{D}^*_m$ satisfies

$$[\,\widetilde{\Lambda}_-(1),\ \widetilde{D}^*_m\,] = -\sqrt{2}(\tilde{c}^*_m\tilde{c}_m - \tilde{c}^*_{\bar{m}}\tilde{c}_{\bar{m}}). \tag{A.4}$$

Therefore, we have

$$\widetilde{\Lambda}_-(1)|m_\nu, m_{\nu-1}, \dots, m_2, m_1\rangle = 0, \tag{A.5}$$

$$0 \le \nu \le \Omega. \tag{A.6}$$

Next, we consider the state

$$|N_0; m_\nu, m_{\nu-1}, \dots, m_2, m_1\rangle = \left(\widetilde{\Lambda}_+(1)\right)^{N_0-\nu}|m_\nu, m_{\nu-1}, \dots, m_2, m_1\rangle. \tag{A.7}$$

The relation (A.7) satisfies

$$\widetilde{\Lambda}_0(1)|N_0; m_\nu, m_{\nu-1}, \ldots, m_2, m_1\rangle = -(\Omega - N_0)|N_0; m_\nu, m_{\nu-1}, \ldots, m_2, m_1\rangle\ , \tag{A.8}$$

i.e.,

$$\widetilde{N}(1)|N_0; m_\nu, m_{\nu-1}, \ldots, m_2, m_1\rangle = N_0|N_0; m_\nu, m_{\nu-1}, \ldots, m_2, m_1\rangle. \tag{A.9}$$

The above is nothing but the relation (4.1). The present eigenvalue problem gives us

$$-(\Omega - \nu) \leq N_0 - \Omega \leq \Omega - \nu. \tag{A.10}$$

Combining with the inequality (A.6), we have the inequality

$$0 \leq \nu \leq \Omega, \qquad \nu \leq N_0 \leq 2\Omega - \nu. \tag{A.11}$$

The above is a possible example of $|N_0\rangle$.

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