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The smallest singular value of certain Toeplitz-related parametric triangular matrices

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Abstract: Let L be the infinite lower triangular Toeplitz matrix with first column $(\mu, a_1, a_2, \ldots, a_p, a_1, \ldots, a_p, \ldots)^T$ and let D be the infinite diagonal matrix whose entries are $1, 2, 3, \ldots$ Let $A := L + D$ be the sum of these two matrices. Bünger and Rump have shown that if $p = 2$ and certain linear inequalities between the parameters μ, a_1, a_2, are satisfied, then the singular values of any finite left upper square submatrix of A can be bounded from below by an expression depending only on those parameters, but not on the matrix size. By extending parts of their reasoning, we show that a similar behaviour should be expected for arbitrary p and a much larger range of values for $\mu, a_1, \ldots, a_p$. It depends on the asymptotics in μ of the l^2-norm of certain sequences defined by linear recurrences, in which these parameters enter. We also consider the relevance of the results in a numerical analysis setting and moreover a few selected numerical experiments are presented in order to show that our bounds are accurate in practical computations.

Keywords: Toeplitz related matrix, triangular matrix, singular value, infinite-dimensional matrix, asymptotics of linear recurrences

MSC: 15B05, 15A18, 39A10, 65Q30

1 Introduction and preliminaries

Given p real numbers $a_1, a_2, \ldots, a_p, \mu$, denote for an integer k by $\overline{k} = k \mod p$ the residue modulo p of k. Define the infinite array $A = (a_{ij})_{i,j=1,2,\ldots}$ by

$$a_{ij} = \begin{cases} \mu + i, & \text{if } i = j, \\ 0, & \text{if } i < j, \\ a_{1+\overline{i-j-1}}, & \text{if } i > j. \end{cases}$$

The left upper $n \times n$ subarrays of A define matrices which we denote by $\mathbf{A}(\mu, a_1, \ldots, a_p, n)$. We will suppress some of the parameters if context allows.

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The lower triangular nature of such matrices follows from the fact that $i < j$ implies $a_{ij} = 0$; that they are mostly Toeplitz follows since if $i-j \neq 0$, the entry a_{ij} depends only on $i-j$. The exception is the main diagonal which is not constant but forms an arithmetic progression. The columns and rows exhibit an almost periodic behaviour due to the periodicity of the map $l \mapsto \bar{l}$.

Example. When $n = 10$ and $p = 4$, we get the matrix

$$\mathbf{A} = \mathbf{A}(\mu, a_1, a_2, a_3, a_4, 10)$$
$$= \begin{pmatrix} \mu+1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_1 & \mu+2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_2 & a_1 & \mu+3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_3 & a_2 & a_1 & \mu+4 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_4 & a_3 & a_2 & a_1 & \mu+5 & 0 & 0 & 0 & 0 & 0 \\ a_1 & a_4 & a_3 & a_2 & a_1 & \mu+6 & 0 & 0 & 0 & 0 \\ a_2 & a_1 & a_4 & a_3 & a_2 & a_1 & \mu+7 & 0 & 0 & 0 \\ a_3 & a_2 & a_1 & a_4 & a_3 & a_2 & a_1 & \mu+8 & 0 & 0 \\ a_4 & a_3 & a_2 & a_1 & a_4 & a_3 & a_2 & a_1 & \mu+9 & 0 \\ a_1 & a_4 & a_3 & a_2 & a_1 & a_4 & a_3 & a_2 & a_1 & \mu+10 \end{pmatrix}$$

In the paper [1] by Bünger and Rump it is shown via elegant reasoning but apparently tailored for the case $p = 2$ that if $\mu > 0$, $1 \le a_1 \le \mu + 3$, and $0 \le a_2 \le a_1 < a_2 + 1$, then for all n the smallest singular value of $\mathbf{A}(\mu, a_1, a_2, n)$ is bounded from below by $\sqrt{\frac{\mu+1}{1+\theta(\mu,a_1,a_2)}}$ where $\theta(\mu, a_1, a_2)$ is an expression in whose definiton n does not enter and which, hence, is independent of n. With this they solved a problem posed by Yoshitaka Watanabe from Kyushu University at the Open Problems session of the workshop Numerical Verification (NIVEA) 2019 in Hokkaido.

The present paper treats in Section 2 the question of a dimension independent lower bound of the singular values for arbitrary p. We transfer the problem more consciously into a question belonging to the asymptotic theory of difference equations. By using recent results of this theory, we hope to be able to show in the near future that the strong hypotheses of our main theorem can in many cases be provably justified; currently we can offer experimental reasons for such a justification.

The evidences gathered are somehow surprising since in the pure Toeplitz setting the minimal singular value can present a remarkable dependency on the matrix size n (see [2–4]). Since the structures studied in this note are encountered in queuing theory, Markov chains, spectral factorizations and the solution of Toeplitz related linear systems, our results can be useful in those areas. Recall that the spectral conditioning is crucial for understanding the achievable precision in the computation of the solution of related linear systems and hence out results are relevant in a numerical analysis context. Therefore in Section 3 numerical experiments are conducted and critically discussed, while Section 4 contains Mathematica© code that allows to experiment conveniently with the sequences defined in the main result (Theorem 1), and Section 5 ends with conclusions and open problems.

2 The Main Result

The section is devoted to the main result regarding lower bounds for the minimal singular value of matrices $\mathbf{A}(\mu, a_1, \ldots, a_p, n)$. Concerning notations, $\sigma_1(X) \ge \sigma_2(X) \ge \cdots \ge \sigma_n(X) \ge 0$ denote the singular values of a square matrix X of size n, $\|\cdot\|_F$, $\|\cdot\|$, $\|\cdot\|_1$, and $\|\cdot\|_\infty$ denote the Frobenius norm, the spectral norm, the l^1 induced matrix norm, and the l^∞ induced matrix norm, respectively, where $\|X\| = \sigma_1(X)$ is the spectral (or l^2 induced) norm,

$$\|X\|_F = \left(\sum_{j,k=1}^{n} |X_{j,k}|^2\right)^{1/2} = \left(\sum_{j=1}^{n} \sigma_j^2(X)\right)^{1/2},$$
$$\|X\|_1 = \max_{k=1,\dots,n} \sum_{j=1}^{n} |X_{j,k}|, \quad \|X\|_\infty = \max_{j=1,\dots,n} \sum_{k=1}^{n} |X_{j,k}|,$$

for X being a square matrix of size n. When it is clear from the context, for a given matrix X and for a proper index j, instead of $\sigma_j(X)$ we will use σ_j.

Our main result concerns the use of the Frobenius norm (see Lemma 1 and Theorem 1), but the other two norms, very popular in a Numerical Analysis setting, are also of interest for the problem under consideration. Before stating and proving the main result, we need a preparatory lemma.

Lemma 1. Given an invertible matrix X of size n we have

$$\sigma_n(X) \geq \|X^{-1}\|_F^{-1}.$$

Proof. It is known that the squared Frobenius norm of a matrix is the sum of the squares of its singular values, see page 421, about 3 centimeters from first text row in [5] ([5, p421c3]). Thus, using the notations above, we have

$$\|X^{-1}\|_F^2 = \sigma_1^2(X^{-1}) + \cdots + \sigma_n^2(X^{-1}) = \frac{1}{\sigma_1^2(X)} + \frac{1}{\sigma_2^2(X)} + \cdots + \frac{1}{\sigma_n^2(X)} \geq \frac{1}{\sigma_n^2(X)},$$

from where the claim follows. □

Theorem 1. Consider nonnegative real numbers $\mu, a_1, \dots, a_p$ and define from these the real sequence $\mathbf{c}(\mu, a_1, \dots, a_p) = (c_m)_{m\geq 1}$ by the equations

$$\begin{aligned} c_1 &= \tfrac{1}{\mu+1} \\ c_j &= \tfrac{1}{\mu+j}\left(-\sum_{l=1}^{j-1} a_l c_{j-l}\right) && \text{for } 2 \leq j \leq p+1 \\ c_m &= \tfrac{1}{\mu+m}\left(-\sum_{l=1}^{p-1} a_l c_{m-l} + (\mu + m - p - a_p)c_{m-p}\right) && \text{for } m \geq p+2. \end{aligned}$$

Assume that the sequence so defined admits an l^2-estimate of the form $\|\mathbf{c}\|^2 = \sum_{l\geq 1} c_l^2 \leq \frac{\theta(\mu)}{(1+\mu)^2},$ with $\mu \mapsto \theta(\mu) = \theta(\mu, a_1, \dots, a_p)$ a nonincreasing function. Then all the singular values of every finite quasi Toeplitz-matrix of the form $\mathbf{A}(\mu, a_1, \dots, a_p, n)$ are (independent of its size) $\geq \sqrt{\frac{\mu}{\theta(\mu)}}$.

Proof. We give a dimension-independent upper bound for the Frobenius Norm of $\mathbf{A}^{-1}$ and this will imply via the previous lemma a lower bound for the smallest singular value.

Define the $n \times n$ matrix $R = (r_{ij})$ by the formula

$$r_{ij} = \begin{cases} \delta_{ij} & \text{if } i \leq p+1 \\ \delta_{ij} - \delta_{i-p,j} & \text{if } i \geq p+2 \ \&\ j \geq 2 \\ 0 & \text{if } i \geq p+2 \ \&\ j = 1. \end{cases}$$

See below for an example of R.

Let $\tilde{\mathbf{A}} = (\tilde{a}_{ij}) = R\mathbf{A}$. We have $\tilde{a}_{ij} = \sum_{v=1}^{n} r_{iv} a_{vj}$. Since for $i \leq p+1$, $r_{iv} = \delta_{iv}$ we see for these i, that $\tilde{a}_{ij} = a_{ij}$. So the first $p+1$ rows of $\tilde{\mathbf{A}}$ coincide with the first $p+1$ rows of $\mathbf{A}$. Now assume $i \geq p+2$. Then, since $r_{i1} = 0$, $\tilde{a}_{ij} = \sum_{v=1}^{n} r_{iv} a_{vj} = \sum_{v=2}^{n} r_{iv} a_{vj} = \sum_{v=2}^{n} (\delta_{iv} - \delta_{i-p,v}) a_{vj} = a_{ij} - a_{i-p,j}$ for $j = 1, 2, \dots, n$.

Thus with obvious notation, we get for $i \geq p+2$ that $\mathrm{row}_i(\tilde{\mathbf{A}}) = \mathrm{row}_i(\mathbf{A}) - \mathrm{row}_{i-p}(\mathbf{A})$.

Consider this case and think of running with j from 1 to n. We have $i - p \geq 2$. Thus for $j = 1, \dots, i-p-1$ we have by definition of a_{ij}, that $a_{ij} - a_{i-p,j} = a_{1+\overline{i-j-1}} - a_{1+\overline{i-p-j-1}} = 0$. For $j = i-p$ we find $a_{i,i-p} - a_{i-p,i-p} = a_{1+\overline{p-1}} - (\mu + i - p) = a_p - \mu - i + p$. For the case $i - p < j < i$ we get $a_{ij} - a_{i-p,j} = a_{1+\overline{i-j-1}}$. For $j = i$ we have $a_{ij} - a_{i-p,j} = a_{ii} - a_{i-p,i} = \mu + i$ and for $j > i$ the difference is $0 - 0 = 0$. Summarizing,

for $i \leq p+1$, $\quad \mathrm{row}_i(\tilde{\mathbf{A}}) = (a_{i-1}, a_{i-2}, \dots, a_1, \mu + i, 0, 0, \dots, 0)$;

$$\text{for } i \geq p+2, \qquad \text{row}_i(\tilde{\mathbf{A}}) = (0, \ldots, 0, \overbrace{a_p - \mu - i + p}^{i-p}, a_{p-1}, \ldots, a_1, \overbrace{\mu + i}^{i}, 0, \ldots 0).$$

Let $e_1 = [1, 0, 0, \ldots, 0]^{\mathsf{T}}$. Note $Re_1 = e_1$ so that also $R^{-1}e_1 = e_1$. Now consider solving the system $\tilde{\mathbf{A}}c = R\mathbf{A}c = e_1$. Then $c = \mathbf{A}^{-1}R^{-1}e_1 = \mathbf{A}^{-1}e_1$. We see that c is the first column of $\mathbf{A}^{-1}$. At the same time the i-th of the equations codified by $\tilde{\mathbf{A}}c = e_1$ is the dot product $\text{row}_i(\tilde{\mathbf{A}}) \cdot c = \delta_{i1}$. Hence we find from the respective rows the following equations.

$$\begin{array}{ll}
\text{row } 1: & (\mu+1)c_1 = 1 \\
\text{row } 2: & a_1c_1 + (\mu+2)c_2 = 0 \\
\text{row } 3: & a_2c_1 + a_1c_2 + (\mu+3)c_3 = 0 \\
\vdots & \vdots \quad \vdots \\
\text{row } p: & a_{p-1}c_1 + a_{p-2}c_2 + \cdots + a_1c_{p-1} + (\mu+p)c_p = 0 \\
\text{row } p+1: & a_pc_1 + a_{p-1}c_2 + \cdots + a_1c_p + c_{p+1}(\mu+p+1) = 0 \\
\text{row } p+2: & (a_p - \mu - 2)c_2 + a_{p-1}c_3 + a_{p-2}c_4 + \cdots + a_2c_p + a_1c_{p+1} + c_{p+2}(\mu+p+2) = 0 \\
\vdots & \vdots \quad \vdots \\
\text{row } m,\ m \geq p+2: & (a_p - \mu - m + p)c_{m-p} + a_{p-1}c_{m-p+1} + a_{p-2}c_{m-p+3} + \cdots + a_1c_{m-1} + (\mu+m)c_m = 0
\end{array}$$

Solving these equations for $c_1, c_2, \ldots, c_n$, respectively, yields precisely the first n equations of the theorem as stated above. Let us write $c(1+\mu)$ for the infinite sequence obtained by the recurrence (so this is just a shorthand for $\mathbf{c}(\mu, a_1, \ldots, a_p)$) and $c(1+\mu)_{1:n}$ for its first n entries. The other columns of $\mathbf{A}^{-1}$ can be obtained essentially in the same fashion: it as an easy consequence of the matrix inversion algorithm that the inverse of the lower right $(n-l) \times (n-l)$ submatrix of an invertible $n \times n$ lower triangular matrix T is the lower right $(n-l) \times (n-l)$ submatrix of the inverse of T. Let us write $\text{lr}_l(T)^{-1} = \text{lr}_l(T^{-1})$ for this fact. Applying the insights obtained for the first column of $\mathbf{A}^{-1}$ to $\text{lr}_1(\mathbf{A})^{-1}$, we see that the second column of $\mathbf{A}^{-1}$ equals $[0, c(\mu+2)_{1:n-1}]^{\mathsf{T}}$. Extending this reasoning we get the following representation of $\mathbf{A}^{-1}$ as a collection of its columns.

$$\mathbf{A}^{-1} = \begin{pmatrix} 0_0 & 0_1 & 0_2 & \ldots & 0_{n-1} \\ c(1+\mu)_{1:n} & c(2+\mu)_{1:n-1} & c(3+\mu)_{1:n-2} & \ldots & c(n+\mu)_{1:1} \end{pmatrix},$$

where 0_j mean j 0s stacked one over another and the $c(i+\mu)_{1:n-i+1}$ should be read as columns. Now the hypothesis on $c = c(1+\mu)$ is that $\|c(1+\mu)\|_2^2 \leq \frac{\theta(\mu)}{(1+\mu)^2}$ for any μ. Therefore,

$$\|\mathbf{A}^{-1}\|_F^2 = \sum_{i=1}^{n} \|c(i+\mu)_{1:n+1-i}\|_2^2 \leq \sum_{i=1}^{n} \|c(i+\mu)\|_2^2 \leq \sum_{i=1}^{n} \frac{\theta(i-1+\mu)}{(i+\mu)^2} \leq \theta(\mu) \sum_{i=1}^{\infty} \frac{1}{(i+\mu)^2}.$$

Now for the last sum we get an estimate by telescoping:

$$\sum_{i=1}^{\infty} \frac{1}{(i+\mu)^2} \leq \sum_{i=1}^{\infty} \frac{1}{(i-1+\mu)(i+\mu)} = \sum_{i=1}^{\infty} \left(\frac{1}{(i-1+\mu)} - \frac{1}{(i+\mu)}\right) = \frac{1}{\mu}.$$

Hence $\|\mathbf{A}^{-1}\|_F^2 \leq \frac{\theta(\mu)}{\mu}$, and by Lemma 1 therefore $\sigma_n(\mathbf{A}) \geq \sqrt{\frac{\mu}{\theta(\mu)}}$. □

Notes: o. An example of a 10×10 matrix R associated to $p = 4$ is the following.

$$R = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1
\end{pmatrix}$$

a. A slightly better estimate than the one above, namely $\sum_{i=1}^{\infty} \frac{1}{(i+\mu)^2} \leq \frac{1}{1+\mu} + \frac{1}{(1+\mu)^2}$ is achievable via the basic case of the Euler-McLaurin Sum formula. It leads to $\sigma_n(\mathbf{A}) \geq \sqrt{\frac{(1+\mu)^2}{\theta(\mu)(2+\mu)}}$.

b. Bünger and Rump give in their paper [1, eq. 24] an inequality $\|\mathbf{A}^{-1}\|_F^2 \leq \frac{1+\theta(\mu)}{1+\mu}$. Due to an oversight there is a small mistake in that paper. The phrase on pdf-page 5 'Then $\hat{A}$ has the same pattern as A with $\hat{\mu} = \mu + j$ instead of μ' should end with '... instead of $\mu+1$.' Correction of the ensuing reasoning leads to the inequality $\|\mathbf{A}^{-1}\|_F^2 \leq \frac{1+\theta(\mu)}{\mu}$ instead of the one given. Also we may observe that the diagonal of $\mathbf{A}^{-1}$ will be $((\mu+1)^{-1}, (\mu+1)^{-1}, \dots, (\mu+n)^{-1})$. We could divide the computation of the l^2-norm of $\mathbf{A}^{-1}$ into the sum of the l^2-norm of the diagonal of $\mathbf{A}$ and the l^2-norm of the remaining entries of $\mathbf{A}^{-1}$. This is in principle what leads to [1, eq. 22]. Roughly then, the $1 + \theta(\mu)$ in [1] corresponds to our $\theta(\mu)$.

c. It is not senseless to speak about the inverse of the infinite array A we have introduced at the beginning of the paper. If one formally applies the inversion algorithm of a tridiagonal matrix to such an infinite array one gets an infinite array that can sensibly be multiplied with A and the sums in the multiplication to be computed would all be finite and give an infinite identity matrix. The upper bound for $\|\mathbf{A}^{-1}\|_F^2$ we computed actually is the upper bound for the Frobenius norm of such an inverse array.

It is of course reasonable to ask to what extent one should believe in the validity of the strong hypotheses of the theorem. The few experiments we did indicate that the hypotheses holds under a wide variety of selections of $\{a_1, \dots, a_p\} \subseteq \mathbb{R}_{\geq 0}$, much larger than what was established for certain in [1]. Below a list of results. The third line, for example, should be read: the sequence $\mu \mapsto (1+\mu)^2\|c(\mu, 1, 0, 1)_{1:107}\|_2^2$ has values 1.40858, 1.00256 for the cases $\mu = 0$ and $\mu = 100$, respectively, and is decreasing as μ runs from 0 to 100. Here for N reasonably large, $\|c(\mu, a_1, \dots, a_p)_{1:N}\|_2^2$ is considered as an approximation for $\|c(\mu, a_1, \dots, a_p)\|_2^2$. It is thus natural to conjecture that indeed even $\|c(\mu, a_1, \dots, a_p)\|_2^2 = \frac{\theta_1(\mu)}{(1+\mu)^2}$ for some strictly decreasing function $\theta_1(\mu)$.

```
 {c{1,1,3}N200, 1.71487, 1.00977, decreasing}
 {c{1,1,4,1,3}N200, 2.67939, 1.01547, decreasing}
 {c{1,0,1}N107, 1.40858, 1.00256, decreasing}
 {c{1,1}N119, 1.28987, 1.00298, decreasing}
{c{1,5,2}N100, 18.881, 1.02895, decreasing}
{c{5,1,2}N100, 92.3826, 1.02958, decreasing}
{c{5,2,1}N100, 191.193, 1.03656, decreasing}
{c{1/19, 3/7, 1/3}N100, 1.03775, 1.00043, decreasing}
{c{1,2,5}N100, 2.8681, 1.01889, decreasing}
{c{1/19, 0.34, 1/3}N100", 1.02779, 1.00033, decreasing}
```

This list can be enlarged in short order by applying the Mathematica© program in Section 4. For integers without decimal points or rationals a_i given as fractions the program works in exact arithmetics. The output is in floating point because at the end a command is used to translate it into this form.

3 Numerical tests and other norms

In this short section we complement the theoretical analysis of Section 2. It is well known that the spectral norm of a normal matrix coincides with the spectral radius $\rho(\cdot)$ and that the spectral radius is bounded from above by any matrix norm which is induced by a vector norm. To see these claims join the observations [5, p417c5, p295c-2, p297c3]. Hence, starting from $\mathbf{A}$ which is not normal (unless it is diagonal), write $\mathbf{A}^{-H}$ for

$(\mathbf{A}^{-1})^*$ and consider $\mathbf{A}^{-1}\mathbf{A}^{-H}$ which is Hermitian and hence normal. Therefore

$$\begin{aligned} \|\mathbf{A}^{-1}\mathbf{A}^{-H}\|_1 &= \|\mathbf{A}^{-1}\mathbf{A}^{-H}\|_\infty \\ &\geq \rho\left(\mathbf{A}^{-1}\mathbf{A}^{-H}\right) \\ &= \frac{1}{\sigma_n^2}, \end{aligned}$$

so that

$$\sigma_n \geq \left(\|\mathbf{A}^{-1}\mathbf{A}^{-H}\|_1\right)^{-1/2} = \left(\|\mathbf{A}^{-1}\mathbf{A}^{-H}\|_\infty\right)^{-1/2},$$

where the latter represents an alternative lowerbound to the minimal singular value.

In [1] Watanabe's problem was tested with a MATLAB program. We generalized this program for different p and $\mu := 100 - \frac{1}{6}$.
The smallest singular value σ_n of $\mathbf{A}$, $\|\mathbf{A}^{-1}\|_F$, and ω are computed for varying dimension by MATLAB, version 2018. Figures (1-8) show that the lower bounds we gave for $\|\mathbf{A}^{-1}\|_F^{-1}$ for different p are asymptotically good. In fact since

$$\sigma_n \geq \|\mathbf{A}^{-1}\|_F^{-1} = \left(\sum_{i=1}^{n}\frac{1}{\sigma_i^2}\right)^{-1/2} = \left(\sum_{i=1}^{n}\frac{1}{\sigma_n^2(\sigma_i/\sigma_n)^2}\right)^{-1/2} = \frac{\sigma_n}{\sqrt{1+(\frac{\sigma_n}{\sigma_1})^2+\cdots+(\frac{\sigma_n}{\sigma_{n-1}})^2}} \geq \omega.$$

It is clear that σ_n is much closer to $\|\mathbf{A}^{-1}\|_F^{-1}$ when σ_{n-1} is much larger than σ_n and that there is a gap between σ_n and $\|\mathbf{A}^{-1}\|_F^{-1}$ when the quantity

$$\triangle = \sqrt{1+(\frac{\sigma_n}{\sigma_1})^2+\cdots+(\frac{\sigma_n}{\sigma_{n-1}})^2}$$

is large. Notice that $1 < \triangle \leq \sqrt{n}$ but the case $\triangle = \sqrt{n}$ cannot be attained for $|a_1|+|a_2|+\cdots+|a_i|>0$ since A is not even normal, while the case $\triangle = \sqrt{n}$ is attained when $\sigma_1 = \sigma_2 = \cdots = \sigma_n = c > 0$, that is when A is a multiple of a unitary matrix: notice that also in the case $|a_1|+|a_2|+\cdots+|a_i| = 0$ the matrix is not a multiple of a unitary matrix. We conclude by observing that all these considerations find a numerical confirmation in Figures (1-8). It is finally worth observing that the numerical lower bound for σ_n related to the l^1 (or l^∞) norms is tighter than that related to the Frobenius norm, at least for moderate sizes (see again Figures (1-8)). A theoretical study of this matter will be the subject of future investigations.

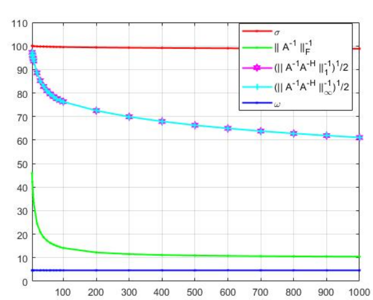

Figure 1: $a_1, a_2 = 7/3, 5/3$ [i=2].

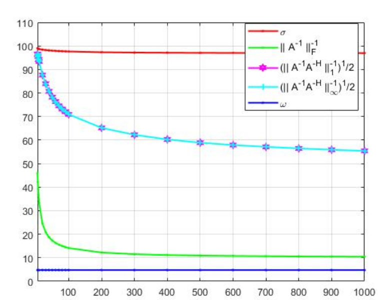

Figure 2: $a_1, a_2, a_3 = 10/3, 1/3, 8/3$ [i=3].

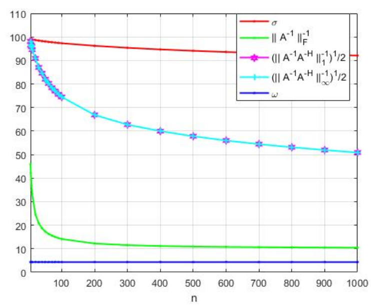

Figure 3: $a_1, a_2, a_3, a_4 = 10/3, 1/3, 2/3, 5/3$ [i=4].

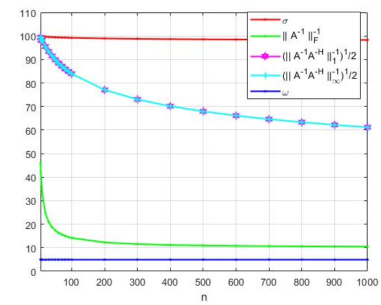

Figure 4: $a_1, a_2, a_3, a_4, a_5 = 20/9, 1/9, 2/9, 1/3, 5/9$ [i=5].

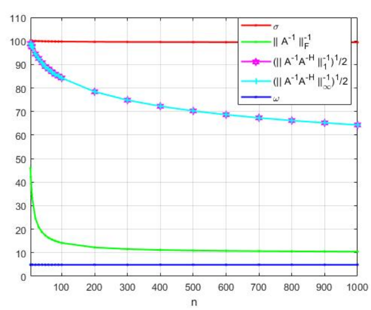

Figure 5: $a_1, a_2, a_3, a_4, a_5, a_6 = 2, 1/2, 2/3, 1, 1/3, 1/3$ [i=6].

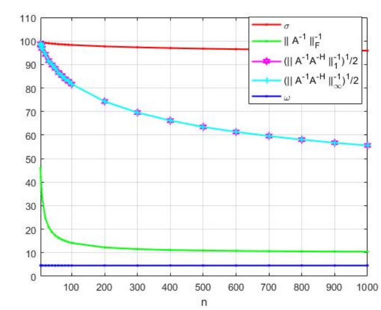

Figure 6: $a_1, a_2, a_3, a_4, a_5, a_6, a_7 = 14/5, 1/5, 2/5, 1, 3/5, 4/5, 1/5$ [i=7].

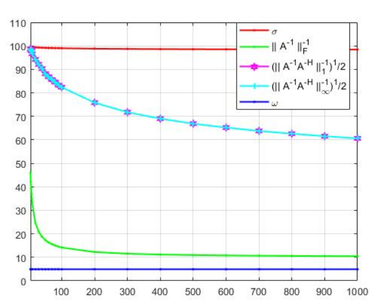

Figure 7: $a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8 = 20/7, 2/7, 4/7, 6/7, 1/7, 5/7, 3/7, 1$ [i=8]
.

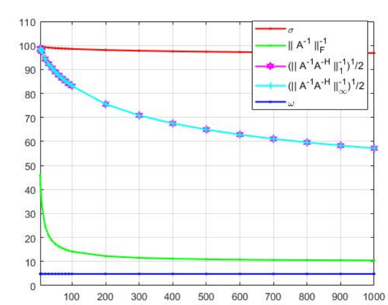

Figure 8: $a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9 = 20/7, 2/7, 4/7, 6/7, 1/7, 5/7, 3/7, 1, 1/7$ [i=9].

4 Some Mathematica© Code

a. The following code computes from an input of reals `a1,a2,. . . ,ap` and a positive integer `nN` given by the user, for every u (meaning μ) from 0 to 100 the value $(1+u)^2 \sum_{i=1}^{nN} c_i^2$. The sequence of these 101 values is collected in the list `ls`. The outer `For`-loop closes after this. The `If[Min . . .]` decides if the list `ls` produced is strictly decreasing. The last line of the code produces a line of the sorts shown and explained towards the end of Section 2. By stripping the program from the outer `For`-loop and ending it after the `While[ . . . ]`

one gets a program allowing to compute, upon inputting `u, lsa, nN`, an individual list `lsc` holding the first `nN` elements of the sequence $\mathbf{c}(\mu, a_1, \ldots, a_p) = (c_1, c_2, \ldots)$ referred in Theorem 1.

```
 lsa={a1,a2, . . .,ap}; nN=. . .;  (*user-defined*)
 ls = {};
For[u = 0, u <= 100, u++,
      p = Length[lsa] ;   ap = Last[lsa];  lsc = {1/(u + 1)};
     For[j=2,j<=p+1,j++,cnew=1/(u+j)*(-Take[lsa,j-1].Reverse[lsc]);AppendTo[lsc,cnew];];
   m = p + 2;
   While[m <= nN,
       cnew=(1/(u+m))(-Take[lsa,p-1].Reverse[Take[lsc,1-p]]+(u+m-p-ap)*lsc[[m-p]]);
       AppendTo[lsc, cnew]; m++]; AppendTo[ls, (1 + u)^2 N[Total[lsc^2]]  ];
  ]
If[Min[Drop[ls-RotateLeft[ls,1],-1]]>0,property="decreasing",property="not-everywhere-decreasing"];
{"c" <> ToString[lsa,InputForm] <> "N" <> ToString[nN], First[ls], Last[ls], property}
```

b. Mathematica commands for the production of $n \times n$ tables A, R and $\tilde{A}$ with periodicity p.

```
{n,p}={ 15 , 3 }  (*user defined*)
matA =Table[Which[i==j,\[Mu]+i,j>i,0,j<i,Subscript[a,1+Mod[i-j-1,p]]],{i,1,n},{j,1,n}];
matR = Table[ Which[i<=p+1,KroneckerDelta[i,j],i>=p+2&&j>=2,
                  KroneckerDelta[i,j]-KroneckerDelta[i-p,j],i>=p+2&&j==1,0],{i,1,n},{j,1,n}];
matAtilde=matR.matA;
```

5 Conclusions

We computed theoretical lowerbounds for the smallest singular value of certain Toeplitz-related parametric triangular matrices with linearly increasing diagonal entries associated with a nonnegative parameter μ. More specifically, the smallest singular value of these matrices is bounded from below by a constant which depends on special entries and on the parameter μ of our matrices and it is independent of the dimension n. The proven result is somehow surprising since in the pure Toeplitz setting the minimal singular value can show a remarkable dependency on the matrix size n. The use of different matrix norms has been considered and some open problems remain concerning the most useful norm in the context of the considered problem. A few selected numerical experiments have been presented and critically discussed, in order to give evidence that our bounds are accurate in practical computations, even if the numerics clearly indicate that the bounds are not sharp and hence there is still room for theoretical improvements.

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