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On the Courant–Fischer theory for Krein spaces

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ABSTRACT

Let $J = I_r \oplus -I_{n-r}, 0 < r < n$. An $n \times n$ complex matrix A is said to be *J-Hermitian* if $JA = A^*J$. An extension of the classical theory of Courant and Fischer on the Rayleigh ratio of Hermitian matrices is stated for *J*-Hermitian matrices. Applications to the theory of small oscilations of a mechanical system are presented.

1. Introduction

For J a Hermitian involutive matrix, that is, $J^* = J, J^2 = I_n$, we consider $\mathbb{C}^n$ endowed with an indefinite inner product $[.,.]$ defined by $[\xi, \eta] = \eta^* J\xi, \xi, \eta \in \mathbb{C}^n$. Let M_n denote the algebra of $n \times n$ complex matrices. A matrix $A \in M_n$ is said to be *J-Hermitian* if $A = A^\#$, where $A^\# = JA^*J$ denotes the *J-adjoint* of A. A matrix $U \in M_n$, is said to be *J-unitary* if $U^\# U = I_n$. For a Hermitian involutive matrix J of signature

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$(r, n-r), 0 < r < n$, the J-unitary matrices form a locally compact group $U_{r,n-r}$, called the *J-unitary group*. The *J-Rayleigh ratio* of a J-Hermitian matrix $A \in M_n$ is the real valued function defined by

$$R_J(x) = \frac{x^*JAx}{x^*Jx}, \quad x^*Jx \neq 0. \tag{1}$$

If $J = I_n$, then (1) reduces to the Rayleigh ratio of a Hermitian matrix. The set of all the possible values for a nonscalar J-Hermitian matrix is unbounded, since the set $\{R_J(x) : x \in \mathbb{C}^n, x^*Jx \neq 0\}$ is neither lower bounded nor upper bounded. However, the set of all values of the J-Rayleigh ratio may be semibounded for some classes of J-Hermitian matrices when we restrict its domain as $X^+ = \{x \in \mathbb{C}^n, x^*Jx > 0\}$ or $X^- = \{x \in \mathbb{C}^n, x^*Jx < 0\}$.

In [1], Ando presented a Löwner inequality of indefinite type, and in [2,11] indefinite versions of well known matrix inequalities were given. These inequalities were obtained in the context of the theory of numerical ranges of operators in spaces with an indefinite inner product, a subject which is being investigated by some authors (see, e.g. [5,6,7,10,12] and the references therein).

This note is organized as follows. In Section 2, an extension of the classical theory of Courant and Fischer on the Rayleigh ratio of Hermitian matrices [3,4,8,9,13,14] is obtained. In Section 3, an application to Hamiltonian dynamics is presented.

2. Courant–Fischer theory for Krein spaces

In [11], the following result was proved.

Lemma 2.1. *Let $A \in M_n$ be J-Hermitian. The set $\{R_J(x) : x \in X^+\}$ is lower bounded (upper bounded) if and only if the set $\{R_J(x) : x \in X^-\}$ is upper bounded (lower bounded). If the former is lower bounded with the greatest lower bound L_1 and the latter is upper bounded with the least upper bound L_2, then these optimal values satisfy $L_2 \leqslant L_1$.*

Necessary and sufficient conditions for a J-Hermitian matrix A to satisfy the above semiboundedness were provided in [11]. To state them, we consider the *generalized eigenspace* $X_\lambda = \{x \in \mathbb{C}^n : (A - \lambda I_n)^n x = 0\}$, and recall that the spectrum of a J-Hermitian matrix is symmetric relatively to the real axis. The following conditions (I) or (II) are necessary for A to satisfy the semiboundedness.

(I) The spectrum $\sigma(A)$ of A is real and $(A - \lambda I_n)x = 0, \forall \lambda \in \sigma(A), \forall x \in X_\lambda$;
(II) The spectrum of A is real, there exists a unique $\lambda_0 \in \sigma(A)$ such that $Ax = \lambda x, \forall \lambda \in \sigma(A)\backslash\{\lambda_0\}, \forall x \in X_\lambda$, and $(A - \lambda_0 I_n)^2 x = 0, \forall x \in X_{\lambda_0}$.

Throughout, we use the notation $\sigma_J^0(A) = \{\lambda_0\}$ in the cases of existence of the exceptional eigenvalue λ_0, otherwise we write $\sigma_J^0(A) = \emptyset$.

If (I) occurs, there exist a set of eigenvectors $\{u_1, \ldots, u_r, u_{r+1}, \ldots, u_n\}$ of A such that $Au_j = \alpha_j u_j (j = 1, \ldots, n)$, where $u_j^* J u_j = 1 (j = 1, \ldots, r)$, $u_j^* J u_j = -1 (j = r+1, \ldots, n)$, and $u_k^* J u_j = 0 (1 \leqslant k \neq j \leqslant n)$, and so A is J-unitarily diagonalizable. We assume that $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_r$ and $\alpha_{r+1} \geqslant \alpha_{r+2} \geqslant \cdots \geqslant \alpha_n$. If A is nonscalar, then $\min\{\alpha_r, \alpha_n\} \neq \max\{\alpha_1, \alpha_{r+1}\}$. Denoting by $\sigma_J^+(A) (\sigma_J^-(A))$ the set of eigenvalues of A, $\lambda \in \mathbb{R}$, such that $Ax = \lambda x$ for some $x \in X^+(X^-)$, we have

$$\sigma_J^+(A) = \{\alpha_1, \ldots, \alpha_r\}, \sigma_J^-(A) = \{\alpha_{r+1}, \ldots, \alpha_n\}.$$

The set $\{R_J(x) : x \in X^+\}$ is semibounded if and only if one of the conditions $\alpha_1 \leqslant \alpha_n$ or $\alpha_{r+1} \leqslant \alpha_r$ is satisfied. If one of these conditions is satisfied, the eigenvalues of A are said to *not interlace*. Otherwise, they are said to *interlace*.

If (II) occurs, the condition for the semiboundedness is more complicated. Under the condition (II), the linear operator A restricted to X_{λ_0} is represented as the direct sum of an operator matrix

$$\begin{pmatrix} \lambda_0 I_s + A_1 & -A_1 \\ A_1 & \lambda_0 I_s - A_1 \end{pmatrix}$$

acting on a Krein space of type (s,s) and a scalar operator $\lambda_0 I_{s_1+s_2}$ acting on a Krein apace of type (s_1,s_2), where A_1 is a positive, or negative, Hermitian matrix. Thus, $\sigma_J^+(A)\backslash\{\lambda_0\}$ is the set of all $\lambda\in\mathbb{R}$ such that $Ax=\lambda x$ for some $x\in X^+ (x\in X^-)$, and $x^*Jy=0$, for $y\in X_{\lambda_0}$. Let $\sigma_J^+(A)\backslash\{\lambda_0\}=\{\alpha_1,\ldots,\alpha_{r-s-s_1}\}, \sigma_J^-(A)\backslash\{\lambda_0\}=\{\alpha_{r+s+s_2+1},\ldots,\alpha_n\}$, be decreasingly ordered. The set $\sigma_J^+(A)$ $(\sigma_J^-(A))$ contains λ_0 if and only if $s_1\geqslant 1 (s_2\geqslant 1)$.

If $\sigma_J^+(A)\backslash\{\lambda_0\}\neq\emptyset$ and $\sigma_J^-(A)\backslash\{\lambda_0\}\neq\emptyset$, the set $\{R_J(x): x\in X^+\}$ is semibounded if and only if one of the conditions (I′) or (II′) is satisfied:

(I′) $\alpha_1\leqslant\lambda_0\leqslant\alpha_n$ and the Hermitian matrix A_1 is negative definite.
(II′) $\alpha_{r+s+s_2+1}\leqslant\lambda_0\leqslant\alpha_{r-s-s_1}$ and the Hermitian matrix A_1 is positive definite.

The eigenvalues of A are said to *not interlace* if one of the above conditions (I′) or (II′) is satisfyed. Otherwise, they are said to *interlace*.

If $\sigma_J^+(A)\backslash\{\lambda_0\}=\emptyset$ and $\sigma_J^-(A)\backslash\{\lambda_0\}=\emptyset$, then the above conditions are relaxed as $-A_1$, or A_1, is positive definite.

If $\sigma_J^+(A)\backslash\{\lambda_0\}\neq\emptyset$ and $\sigma_J^-(A)\backslash\{\lambda_0\}=\emptyset$, then the conditions (I′), (II′) are relaxed as the following:

(I″) $\alpha_1\leqslant\lambda_0$ and the Hermitian matrix A_1 is negative definite.
(II″) $\lambda_0\leqslant\alpha_{r-s-s_1}$ and the Hermitian matrix A_1 is positive definite.

If $\sigma_J^-(A)\backslash\{\lambda_0\}\neq\emptyset$ and $\sigma_J^+(A)\backslash\{\lambda_0\}=\emptyset$, then:

(I‴) $\lambda_0\leqslant\alpha_n$ and the Hermitian matrix A_1 is negative definite.
(II‴) $\lambda_0\leqslant\alpha_{r+s+s_2+1}$ and the Hermitian matrix A_1 is positive definite.

For an arbitrary linear subspace S of $\mathbb{C}^n$, let $S^+=\{x\in S: x^*Jx>0\}$ and $S^-=\{x\in S: x^*Jx<0\}$.
The following results were obtained in [2,11].

Theorem 2.1. *Let $J=I_r\oplus -I_{n-r}, 0<r<n$, and let $A\in M_n$ be J-Hermitian with noninterlacing eigenvalues.*
(I$_0$) *The case $\sigma_J^0(A)=\emptyset$. Let $\sigma_J^+(A)=\{\alpha_1,\ldots,\alpha_r\}$ and $\sigma_J^-(A)=\{\alpha_{r+1},\ldots,\alpha_n\}$ be decreasingly ordered. The following holds*:

(a) *If $\alpha_n>\alpha_1$, then $\max_{x^*Jx=1} x^*JAx=\alpha_1, \min_{x^*Jx=-1} x^*JAx=\alpha_n$, and conversely.*
(b) *If $\alpha_r>\alpha_{r+1}$, then $\max_{x^*Jx=-1} x^*JAx=\alpha_{r+1}, \min_{x^*Jx=1} x^*JAx=\alpha_r$, and conversely*.

(II′) *The case $\sigma_J^0(A)=\{\lambda_0\}$. Let $\sigma_J^+(A)\backslash\{\lambda_0\}=\{\alpha_1,\ldots,\alpha_{r-s-s_1}\}$, $\sigma_J^-(A)\backslash\{\lambda_0\}=\{\alpha_{r+s+s_2+1},\ldots,\alpha_n\}$ be decreasingly ordered. Let the multiplicities of the eigenvalue $\lambda_0\in\sigma_J^+(A)$ and $\lambda_0\in\sigma_J^-(A)$ be s_1 and s_2, respectively. Let the pure part of A on X_{λ_0} be acting on a Krein space of type (s,s). The following holds:*

(a′) *Let $\alpha_n\geqslant\lambda_0\geqslant\alpha_1$.*
If $s_1\geqslant 1$, then $\max_{x^*Jx=1}x^*JAx=\lambda_0$.
If $s_1=0$, then $R_J(x)<\sup_{y^*Jy=1}y^*JAy=\lambda_0, \forall x\in X^+$.
If $s_2\geqslant 1$, then $\min_{x^*Jx=-1}(-x^*JAx)=\lambda_0$.
If $s_2=0$, then $\inf_{y^*Jy=-1}(-y^*JAy)=\lambda_0<R_J(x), \forall x\in X^-$.
(b′) *Let $\alpha_{r-s-s_1}\geqslant\lambda_0\geqslant\alpha_{r+s+s_2+1}$.*
If $s_2\geqslant 1$, then $\max_{x^*Jx=-1}(-x^*JAx)=\lambda_0$.
If $s_2=0$, then $R_J(x)<\sup_{y^*Jy=-1}(-y^*JAy)=\lambda_0, \forall x\in X^-$.
If $s_1\geqslant 1$, then $\min_{x^*Jx=1}x^*JAx=\lambda_0$.
If $s_1=0$, then $\inf_{y^*Jy=1}y^*JAy=\lambda_0<R_J(x), \forall x\in X^+$.

Next we consider the J-Rayleigh ratio of a vector x ranging over an arbitrary $(n-i+1)$- dimensional subspace of $\mathbb{C}^n$, and we extend Theorem 2.1 to subspaces. We investigate how far this J-Rayleigh ratio is from the bounds obtained in the above mentioned theorem.

Theorem 2.2. *Let $J = I_r \oplus -I_{n-r}, 0 < r < n$, and let $A \in M_n$ be a J-unitarily diagonalizable J-Hermitian matrix with noninterlacing eigenvalues satisfying $\alpha_n \geqslant \alpha_1$ or $\alpha_r \geqslant \alpha_{r+1}$.*

(a) *If $\alpha_n > \alpha_1$ or $\alpha_r > \alpha_{r+1}$, then the sets $\{R_J(x) : x \in S^+\}$ and $\{R_J(x) : x \in S^-\}$ are closed for an arbitrary linear subspace S of $\mathbb{C}^n$.*
(b) *If $\{u_1, \ldots, u_r, u_{r+1}, \ldots, u_n\}$ is a J-orthonormal system of eigenvectors of A with associated eigenvalues $\alpha_1, \ldots, \alpha_r, \alpha_{r+1}, \ldots, \alpha_n$ and S is the linear subspace spanned by $\{u_{i_1}, \ldots, u_{i_s}, u_{i_{s+1}}, \ldots, u_{i_m}\}$, then the sets $\{R_J(x) : x \in S^+\}$ and $\{R_J(x) : x \in S^-\}$ are closed under the condition $\alpha_n \geqslant \alpha_1$ or $\alpha_r \geqslant \alpha_{r+1}$.*

Proof. (a) Let $\alpha_n > \alpha_1$ and consider a sequence of vectors x_m in S such that $x_m^* J x_m = -1$ and

$$R_J(x_m) = -x_m^* JA x_m \to R_0 \geqslant \alpha_n \tag{2}$$

as $m \to \infty$. We observe that the last inequality in (2) is a consequence of Theorem 2.1 (a). Expressing x_m in the J-orthonormal basis $\{u_1, \ldots, u_r, u_{r+1}, \ldots, u_n\}$ as $x_m = \sum_{j=1}^n a_j^{(m)} u_j$, we *claim* that the set

$$\{a_j^{(m)} \in \mathbb{C} : m = 1, 2, 3, \ldots\} \tag{3}$$

is bounded for each $1 \leqslant j \leqslant n$. Assuming that the claim is proved, by taking a subsequence of (3) we may conclude that there exists a vector $x_\infty = \sum_{j=1}^n a_j^{(\infty)} u_j \in S$ satisfying $|a_j^{(m)} - a_j^{(\infty)}| \to 0$ as $m \to \infty$. Thus, the vector $x_\infty \in S$ satisfies $x_\infty^* J x_\infty = -1$ and $-x_\infty^* JA x_\infty = R_0$.

We prove the claim by *contradiction*. Indeed, suppose that (3) is unbounded and assume that the sequence $(m_p)_{p=1}^\infty$ satisfies

$$\sum_{k=r+1}^{n} \left|a_k^{(m_p)}\right|^2 = \sum_{k=1}^{r} \left|a_k^{(m_p)}\right|^2 + 1 \to \infty$$

as $p \to \infty$. Then, we have

$$\begin{aligned} -x_{m_p}^* JA x_{m_p} &= \sum_{k=r+1}^{n} \alpha_k \left|a_k^{(m_p)}\right|^2 - \sum_{k=1}^{r} \alpha_k \left|a_k^{(m_p)}\right|^2 \\ &\geqslant \alpha_n + \alpha_n \sum_{k=1}^{r} \left|a_k^{(m_p)}\right|^2 - \alpha_1 \sum_{k=1}^{r} \left|a_k^{(m_p)}\right|^2 \\ &= \alpha_n + (\alpha_n - \alpha_1) \sum_{k=1}^{r} \left|a_k^{(m_p)}\right|^2 \to \infty \end{aligned}$$

as $p \to \infty$, which contradicts (2). Thus, (3) is bounded. The case $\alpha_r > \alpha_{r+1}$ is treated similarly.

(b) In this case, $A(S) \subset S$ and by considering the restriction of A to S, we may assume that $S = \mathbb{C}^n$. Let $\alpha_n = \alpha_1 > \alpha_r$or $\alpha_{r+1} > \alpha_n = \alpha_1$. We show that under the condition $\alpha_{r+1} > \alpha_n$, we necessarily have

$$\{R_J(x) : x \in X^+\} = (-\infty, \alpha_1], \quad \{R_J(x) : x \in X^-\} = [\alpha_n, \infty).$$

In fact, for $w(t) = (\cosh tu_1 + \sinh tu_{r+1})$ we get $w(t)^* J w(t) = 1$ and

$$w(t)^* JA w(t) = \alpha_1 - (\alpha_{r+1} - \alpha_1) \sinh^2 t,$$

$(0 \leqslant t < \infty)$. For $v(t,s) = (\sinh tu_1 + \cosh t(\sqrt{s} u_{r+1} + \sqrt{1-s} u_n))$ we find $v(t,s)^* J v(t,s) = -1$ and

$$v(t,s)^* JA v(t,s) = -(1-s)\alpha_1 - s(\alpha_{r+1} + (\alpha_{r+1} - \alpha_n) \sinh^2 t),$$

$(0 \leqslant t < \infty, 0 \leqslant s \leqslant 1)$. Thus, we obtain the desired relation. □

We treat three classes of J-Hermitian matrices with noninterlacing eigenvalues:

(I_0): $\sigma_J^0(A) = \emptyset$ and $\alpha_n > \alpha_1$ or $\alpha_r > \alpha_{r+1}$.
(I_1): $\sigma_J^0(A) = \emptyset$, A is nonscalar and $\alpha_n = \alpha_1$ or $\alpha_r = \alpha_{r+1}$.
(II_1): $\sigma_J^0(A) = \{\lambda_0\}$ and the pure part of A on X_{λ_0} acts on a Krein space of type (s,s) with $s \geqslant 1$, being $s_1 \geqslant 0$ and $s_2 \geqslant 0$ the multiplicities of $\lambda_0 \in \sigma_J^+(A)$ and $\lambda_0 \in \sigma_J^-(A)$, respectively.

Sometimes, it is convenient to consider the class (I_1) as a degenerate class of (II_1) with $\lambda_0 = \alpha_n = \alpha_1$ or $\lambda_0 = \alpha_r = \alpha_{r+1}, s = 0, s_1 \geqslant 1, s_2 \geqslant 1$. We represent by ($II'$) the union of ($I_1$) and ($II_1$).

Theorem 2.3. *For $J = I_r \oplus -I_{n-r}, 0 < r < n$, let $A \in M_n$ be a J-Hermitian matrix with noninterlacing eigenvalues. Let $S_i (1 \leqslant i \leqslant n)$ be an arbitrary $(n-i+1)$-dimensional linear subspace of $\mathbb{C}^n$.*

(I_0) *The case $\sigma_J^0(A) = \emptyset$. Let $\sigma_J^+(A) = \{\alpha_1, \ldots, \alpha_r\}$, $\sigma_J^-(A) = \{\alpha_{r+1}, \ldots, \alpha_n\}$ be decreasingly ordered. The following holds*:

(a) *Let $\alpha_n > \alpha_1$.*
If $i \leqslant n-r$, then $\min_{x \in S_i^-} R_J(x) \leqslant \alpha_{n-i+1}$, and $\max_{S_i} \min_{x \in S_i^-} R_J(x) = \alpha_{n-i+1}$.
If $i \leqslant r$, then $\max_{x \in S_i^+} R_J(x) \geqslant \alpha_i$ and $\min_{S_i} \max_{x \in S_i^+} R_J(x) = \alpha_i$.

(b) *Let $\alpha_r > \alpha_{r+1}$.*
If $i \leqslant r$, then $\min_{x \in S_i^+} R_J(x) \leqslant \alpha_{r-i+1}$ and $\max_{S_i} \min_{x \in S_i^+} R_J(x) = \alpha_{r-i+1}$.
If $i \leqslant n-r$, then $\max_{x \in S_i^-} R_J(x) \geqslant \alpha_{r+i}$ and $\min_{S_i} \max_{x \in S_i^-} R_J(x) = \alpha_{r+i}$.

(II') *The case $\sigma_J^0(A) = \{\lambda_0\}$ or $\sigma_J^0(A) = \emptyset$ and $\sigma_J^+(A) \cap \sigma_J^-(A) = \{\lambda_0\}$. Let $\sigma_J^+(A) \backslash \{\lambda_0\} = \{\alpha_1, \cdots, \alpha_{r-s-s_1}\}$, $\sigma_J^-(A) \backslash \{\lambda_0\} = \{\alpha_{r+s+s_2+1}, \cdots, \alpha_n\}$, be decreasingly ordered. Let A restricted to X_{λ_0} be the direct sum of $\lambda_0 I_{s_1+s_2}$ on a Krein space of type (s_1, s_2) and a pure nondiagonalizable part on a Krein space of type (s,s). The following holds:*

(a') *Let $\alpha_n \geqslant \lambda_0 \geqslant \alpha_1$.*
If $s + s_2 + 1 \leqslant i \leqslant n-r$, then $\inf_{x \in S_i^-} R_J(x) \leqslant \alpha_{n-i+s+s_2+1}$, and $\max_{S_i} \inf_{x \in S_i^-} R_J(x) = \alpha_{n-i+s+s_2+1}$.
If $s \geqslant 1$ and $s_2 + 1 \leqslant i \leqslant s_2 + s$, then $\inf_{x \in S_i^-} R_J(x) = \lambda_0$, and $\sup_{S_i} \inf_{x \in S_i^-} R_J(x) = \lambda_0$.
If $s_2 \geqslant 1$ and $1 \leqslant i \leqslant s_2$, then $\min_{x \in S_i^-} R_J(x) = \lambda_0$, and $\max_{S_i} \min_{x \in S_i^-} R_J(x) = \lambda_0$.
If $s + s_1 + 1 \leqslant i \leqslant r$, then $\sup_{x \in S_i^+} R_J(x) \geqslant \alpha_{i-s-s_1}$, and $\min_{S_i} \sup_{x \in S_i^+} R_J(x) = \alpha_{i-s-s_1}$.
If $s \geqslant 1$ and $s_2 + 1 \leqslant i \leqslant s_1 + s$, then $\sup_{x \in S_i^+} R_J(x) = \lambda_0$, and $\inf_{S_i} \sup_{x \in S_i^+} R_J(x) = \lambda_0$.
If $s_1 \geqslant 1$ and $1 \leqslant i \leqslant s_1$, then $\max_{x \in S_i^+} R_J(x) = \lambda_0$, and $\min_{S_i} \max_{x \in S_i^+} R_J(x) = \lambda_0$.

(b') *Let $\alpha_{r-s-s_1} \geqslant \lambda_0 \geqslant \alpha_{r+s+s_2+1}$.*
If $s + s_1 + 1 \leqslant i \leqslant r$, then $\inf_{x \in S_i^+} R_J(x) \leqslant \alpha_{r-i+1}$, and $\max_{S_i} \inf_{x \in S_i^+} R_J(x) \leqslant \alpha_{r-i+1}$.
If $s \geqslant 1$ and $s_1 + 1 \leqslant i \leqslant s_1 + s$, then $\inf_{x \in S_i^+} R_J(x) = \lambda_0$, and $\sup_{S_i} \inf_{x \in S_i^+} R_J(x) = \lambda_0$.
If $s_1 \geqslant 1$ and $1 \leqslant i \leqslant s_1$, then $\min_{x \in S_i^+} R_J(x) = \lambda_0$, and $\max_{S_i} \min_{x \in S_i^+} R_J(x) = \lambda_0$.
If $s + s_2 + 1 \leqslant i \leqslant n-r$, then $\sup_{x \in S_i^-} R_J(x) \geqslant \alpha_{r+i}$ and $\min_{S_i} \sup_{x \in S_i^-} R_J(x) = \alpha_{r+i}$.
If $s \geqslant 1$ and $s_2 + 1 \leqslant i \leqslant s_2 + s$, then $\sup_{x \in S_i^-} R_J(x) = \lambda_0$, and $\inf_{S_i} \sup_{x \in S_i^-} R_J(x) = \lambda_0$.
If $s_2 \geqslant 1$ and $1 \leqslant i \leqslant s_2$, then $\max_{x \in S_i^-} R_J(x) = \lambda_0$, and $\min_{S_i} \max_{x \in S_i^-} R_J(x) = \lambda_0$.

Proof. (I_0) The case $\sigma_J^0(A) = \emptyset$. Let $\{u_1, \ldots, u_n\}$ be a standard J-orthonormal system of eigenvectors of A associated with the eigenvalues $\alpha_1, \ldots, \alpha_n$. We prove (a). Suppose that $i \leqslant n-r$. Let T_i be the linear space spanned by the set of vectors $\{u_{n-i+1}, \ldots, u_n\}$, where $r + 1 \leqslant n - i + 1$. There exists a nonzero vector $u \in \mathbb{C}^n$ belonging to S_i and T_i, because $\dim S_i + \dim T_i = (n-i+1) + i = n+1$. Since $u \in T_i$, it follows that there exist $a_j \in \mathbb{C}, j = 1, \ldots, i$, such that $u = \sum_{j=n-i+1}^{n} a_j u_j$. Since $i \leqslant n-r$, all the vectors $u_{n-i+1}, \ldots, u_n$ have negative J-norm. Therefore, $u^* J u < 0$. We clearly have

$$\frac{u^*JAu}{u^*Ju} = \frac{-\sum_{k=n-i+1}^{n}\alpha_j|a_j|^2}{-\sum_{k=n-i+1}^{n}|a_j|^2} = \frac{\sum_{k=n-i+1}^{n}\alpha_j|a_j|^2}{\sum_{k=n-i+1}^{n}|a_j|^2}.$$

Thus, since we are assuming that $\alpha_n > \alpha_1$

$$\alpha_n \leqslant R_J(u) \leqslant \alpha_{n-i+1}.$$

Recalling that u also belongs to S_i, from Lemma 2.2 (I_0) it follows that

$$\inf_{x\in S_i^-} R_J(x) \leqslant R_J(x) \leqslant \alpha_{n-i+1}$$

and by Theorem 2.1 the greatest lower bound of $R_J(x)$ is attained at some vector $x \in S_i^-$.

To prove that $\max_{S_i}\min_{x\in S_i^-} R_J(x) = \alpha_{n-i+1}$, it suffices to show the existence of an $(n-i+1)$-dimensional subspace such that $\min R_J(x)$, when x ranges over all nonzero vectors of this subspace, is equal to α_{n-i+1}. Consider the linear space V_i spanned by the vectors $u_1,\ldots,u_r,u_{r+1},\ldots,u_{n-i+1}$. For any $u \in V_i$ such that $u^*Ju < 0$, there are complex numbers $c_1,\ldots,c_{n-i+1}$ such that $u = \sum_{k=1}^{n-i+1} c_k u_k$. We have

$$\frac{u^*JAu}{u^*Ju} = \frac{\sum_{k=1}^{n-i+1} c_k^* c_k \alpha_k j_k}{\sum_{k=1}^{n-k+1} c_k^* c_k j_k} \geqslant \alpha_{n-i+1}.$$

Hence $\min_{u\in V_i} R_J(u) = \alpha_{n-i+1}$, being the minimum attained when $c_{n-i+1} \neq 0$ and $c_1 = c_2 = \cdots = c_{n-i+2} = 0$. That is, the minimizing vector is an eigenvector of A associated with α_{n-i+1}.

Now, let $i \leqslant r$. Considering the i-dimensional linear space T_i spanned by $u_1,\ldots,u_i$, we may conclude that there exists a vector $u' \in \mathbb{C}^n$ belonging simultaneously to S_i and T_i, because $\dim S_i + \dim T_i = (n-r+i)+i = n+1$. Since $i \leqslant r$, we have $u' = \sum_{j=1}^{i} a_j' u_j$, for a_j' not all zero $(j = 1,\ldots,i)$. So $u'^*Ju' > 0$, because all the vectors $u_1,\ldots,u_i$ have positive J-norm. We easily find

$$\frac{u'^*JAu'}{u'^*Ju'} = \frac{\sum_{j=1}^{i}\alpha_j|a_j'|^2}{\sum_{j=1}^{n}|a_j'|^2}.$$

Hence, under the assumption $\alpha_n > \alpha_1$, we obtain $\alpha_i \leqslant R_J(u') \leqslant \alpha_1$. Recalling that u' belongs to S_i, from Theorem 2.1 (I_0) it follows that

$$\sup_{x\in S_i^+} R_J(x) \geqslant R_j(u') \geqslant \alpha_i.$$

By Theorem 2.2, it can be shown that the minimum is attained at a certain $x_i \in S_i^+$.

The other statement is proved similarly.

(b) The proof follows analogous steps to (a).

(II') The proof is similar to that of (I_0). □

In the next theorem we denote by $R_J^{A+B}(x)$ the J-Rayleigh ratio associated with $A+B$.

Proposition 2.4. *Let $J = I_r \oplus -I_{n-r}, 0 < r < n$, and let $A, B \in M_n$ be J-unitarily diagonalizable J-Hermitian matrices with noninterlacing eigenvalues, $\alpha_1 \geqslant \cdots \geqslant \alpha_r \in \sigma_J^+(A), \alpha_{r+1} \geqslant \cdots \geqslant \alpha_n \in \sigma_J^-(A)$ and $\beta_1 \geqslant \cdots \geqslant \beta_r \in \sigma_J^+(B), \beta_{r+1} \geqslant \cdots \geqslant \beta_n \in \sigma_J^-(B)$. Let S_i be an $(n-i+1)$-dimensional linear subspace of $\mathbb{C}^n$.*

(a) *Let $\alpha_n > \alpha_1$ and $\beta_n > \beta_1$. Then $A+B$ is J-unitarily diagonalizable and the following holds.*
If $n-i \geqslant r$, then $\max_{S_i}\min_{x\in S_i^-} R_J^{A+B}(x) \geqslant \alpha_{n-i+1} + \beta_n$.
If $i \leqslant r$, then $\min_{S_i}\max_{x\in S_i^+} R_J^{A+B}(x) \leqslant \alpha_i + \beta_1$.

(b) *Let $\alpha_r > \alpha_{r+1}$ and $\beta_r > \beta_{r+1}$. Then $A+B$ is J-unitarily diagonalizable and the following holds.*
If $i \leqslant r$, then $\max_{S_i}\min_{x\in S_i^+} R_J^{A+B}(x) \geqslant \alpha_{r-i+1} + \beta_r$.
If $r+i \leqslant n$, then $\min_{S_i}\max_{x\in S_i^-} R_J^{A+B}(x) \leqslant \alpha_{r+i} + \beta_{r+1}$.

Proof. (a) Let $\alpha_n > \alpha_1$ and $\beta_n > \beta_1$.

For any x, y such that $x^*Jx = 1, y^*Jy = -1$ we get

$$-y^*J(A+B)y = -y^*JAy - y^*JBy \geqslant \alpha_n + \beta_n$$
$$> \alpha_1 + \beta_1 \geqslant x^*JAx + x^*JBx = x^*J(A+B)x$$

and so the J-Hermitian matrix $A+B$ is J-unitarily diagonalizable.

If $n - i \geqslant r$, by Theorem 2.3 (a) we have

$$\max_{S_i} \min_{x \in S_i^-} \frac{x^*J(A+B)x}{x^*Jx} \geqslant \max_{S_i} \min_{x \in S_i^-} \frac{x^*JAx}{x^*Jx} + \beta_n = \alpha_{n-i+1} + \beta_n.$$

Analogously, if $i \leqslant r$

$$\min_{S_i} \max_{x \in S_i^+} R_J^{A+B}(x) \leqslant \alpha_i + \beta_1$$

and (a) follows. The proof of (b) is similar. □

For $i = 1, \ldots, n, \lambda_i(X)$ denotes the eigenvalues of $X \in M_n$ decreasingly ordered.

Theorem 2.5. *Suppose that A, B are J-Hermitian matrices with noninterlacing eigenvalues. Let $\sigma_J^+(A)\backslash\{\lambda_0\} = \{\alpha_1, \ldots, \alpha_{r-s-s_1}\}$, $\sigma_J^-(A)\backslash\{\lambda_0\} = \{\alpha_{r+s+s_2+1}, \ldots, \alpha_n\}$ and $\sigma_J^+(B)\backslash\{\gamma_0\} = \{\beta_1, \cdots, \beta_{r-t-t_1}\}$, $\sigma_J^-(B)\backslash\{\gamma_0\} = \{\beta_{r+t+t_2+1}, \ldots, \beta_n\}$, be decreasingly ordered and satisfy*

$$\alpha_n > \alpha_1, \quad \beta_n > \beta_1.$$

Then all the eigenvalues of $A + B$ are real and the following inequalities hold

$$\sum_{j=n-r+1}^{n-r+k} \lambda_j(A+B) \leqslant \sum_{j=n-r+1}^{n-r+k} (\lambda_j(A) + \lambda_j(B)), \quad k = 1, \ldots, r$$

and

$$\sum_{j=1}^{k-r} \lambda_j(A+B) + \sum_{j=n-r+1}^{n} \lambda_j(A+B)$$
$$\leqslant \sum_{j=1}^{k-r} (\lambda_j(A) + \lambda_j(B)) + \sum_{j=n-r+1}^{n} (\lambda_j(A) + \lambda_j(B)), \quad k = r+1, \ldots, n.$$

Proof. To prove the theorem, we recall the extremal representation obtained in Theorem 3.1 of [2]. Suppose that A is a J-diagonalizable J-Hermitian matrix with noninterlacing eigenvalues $\alpha_1 \geqslant \cdots \geqslant \alpha_r$ in $\sigma_J^+(A)$ and $\alpha_{r+1} \geqslant \cdots \geqslant \alpha_n$ in $\sigma_J^-(A)$ satisfying $\alpha_n > \alpha_1$.

Let k be an arbitrary natural number satisfying $1 \leqslant k \leqslant n$. Then there exists a J-orthonormal system of vectors $\{u_1, \ldots, u_r, u_{r+1}, \ldots, u_k\}$ such that the form

$$F_k(A; u_1, \ldots, u_r, u_{r+1}, \ldots, u_k) := \sum_{j=1}^{r} u_j^* JAu_j - \sum_{j=r+1}^{k} u_j^* JAu_j$$

attains the maximum $\lambda_{n-r+1}(A) + \cdots + \lambda_{n-r+k}(A)$ at this system when $k \leqslant r$. If $k > r$, the maximum is replaced by $\lambda_1(A) + \cdots + \lambda_{k-r}(A) + (\lambda_{r+1}(A) + \cdots + \lambda_n(A))$.

Next we use a perturbative method. We consider the case $\sigma_J^0(A) \neq \emptyset$ or $\sigma_J^0(B) \neq \emptyset$. The J-Hermitian matrix

$$\begin{pmatrix} \lambda_0 + 1 & -1 \\ 1 & \lambda_0 - 1 \end{pmatrix},$$

which is nondiagonalizable under a J-unitary transformation, is approximated by the J-Hermitian matrix

$$\begin{pmatrix} \lambda_0+1+\epsilon & -1 \\ 1 & \lambda_0-1-\epsilon \end{pmatrix},$$

where $\epsilon > 0$, which is J-unitarily diagonalizable. We also perturbe the eigenvalues of A and B so that they satisfy the condition (I), having in mind that the eigenvalues of a matrix depend continuously on its entries. So we may assume that A, B are J-unitarily diagonalizable and satisfy $\alpha_n > \alpha_1$ and $\beta_n > \beta_1$. Then from the inequality

$$\begin{aligned} &\max_{u_1,\ldots,u_k} F_k(A+B; u_1,\ldots,u_r,u_{r+1},\ldots,u_k) \\ &\quad\leqslant \max_{u_1,\ldots,u_k} F_k(A; u_1,\ldots,u_r,u_{r+1},\ldots,u_k) + \max_{u_1,\ldots,u_k} F_k(B; u_1,\ldots,u_r,u_{r+1},\ldots,u_k) \end{aligned}$$

the desired inequality follows. □

Theorem 2.6. *Let $J = I_r \oplus -I_{n-r}, 0 < r < n$, and let $A \in M_n$ be a J-Hermitian matrix with noninterlacing eigenvalues. Then any principal submatrix A' of A has real spectrum and its eigenvalues do not interlace. Moreover, if A satisfies the condition* (I′) *or* (II′) *and A' acts on a Krein space of type $(s, n-i+1-s)$, then the following inequalities hold*:

$$\lambda_{r-i+1-t}(A) \geqslant \lambda_{s-t}(A')$$

for $s-t \geqslant 1, r-i+1-t \geqslant 1$, and

$$\lambda_{s+1+t}(A') \geqslant \lambda_{r+i+t}(A),$$

for $s+1+t \leqslant n-i+1, r+i+t \leqslant n$.

Proof. By using a perturbative method, we may assume that A is J-unitarily diagonalizable and satisfies $\alpha_r > \alpha_{r+1}$. As an operator defined on a nondegenerate subspace of $\mathbb{C}^n$, the submatrix A' of A is J-Hermitian. Since the J-Rayleigh ratio relative to A' satisfies the semiboundedness, the eigenvalues of A' are real. The J-Rayleigh ratio relative to A' is a restriction of the J-Rayleigh ratio of A, so the inequalities $\lambda_s(A') \geqslant \lambda_r(A)$ and $\lambda_{s+1}(A') \leqslant \lambda_{r+1}(A)$ hold. Thus, the eigenvalues of A' do not interlace.

Let $w_1,\ldots,w_{s-t},w_{s+1},\ldots,w_{n-i+1}$ be the J-orthonormal eigenvectors of A' associated with the eigenvalues $\lambda_1(A'),\ldots,\lambda_{s-t}(A'),\lambda_{s+1}(A'),\ldots,\lambda_{n-i+1}(A')$, respectively. Consider the $(n-i+1-t)$-dimensional linear subspace S_i generated by these eigenvectors. By Theorem 2.3 (I_0), we have

$$\lambda_{s-t}(A') = \min_{x\in S_i^+} R_J(x) \leqslant \lambda_{r+1-i-t}(A).$$

The theorem easily follows using similar arguments.

Theorem 2.7. *If A is a J-Hermitian matrix, then $R_J(x)$ for any $x \in X^+ (x \in X^-)$ has a stationary value with respect to x at an eigenvector x_0 associated with a real eigenvalue α_0, $x_0^*Jx_0 = 1\,(x_0^*Jx_0 = -1)$ and $R_J(x_0) = \alpha_0$.*

Proof. Let $x = \Re x + i\Im x \in X^+, \Re x = (\xi_1,\ldots,\xi_n)^T = \sum_{k=1}^n \xi_k e_k^T$ and $\Im x = (\eta_1,\ldots,\eta_n)^T = \sum_{k=1}^n \eta_k e_k^T$, being $\{e_1,\ldots,e_n\}$ the standard basis of $\mathbb{C}^n$. Viewing $R_J(x)$ as a real valued function of the $2n$-independent real variables $\xi_1,\ldots,\xi_n$ and $\eta_1,\ldots,\eta_n$, we write $R_J(x) = R_J(\xi_1,\ldots,\xi_n,\eta_1,\ldots,\eta_n)$. Consider the bilinear form

$$\Phi(\xi_1,\ldots,\xi_n,\eta_1,\ldots,\eta_n) = x^*JAx - \tau x^*Jx,$$

where τ is a real Lagrange multiplier fixing the norm of x, that is, $x^*Jx = 1$. It is equivalent to require that the partial derivatives of R_J, or of Φ, vanish. We easily find that

$$\frac{\partial \Phi}{\partial \xi_k} = e_k^T JAx - \tau e_k^T Jx + x^* JAe_K - \tau x^* Je_k = 0,$$
$$\frac{\partial \Phi}{\partial \eta_k} = -ie_k^T JAx + i\tau e_k^T Jx + ix^* JAe_K - i\tau x^* Je_k = 0, \quad k = 1, \ldots, n.$$

Thus, there exist $x = x_0$ and $\tau = \alpha_0$ such that

$$Ax_0 = \alpha_0 x_0, \quad x_0^* JA = \alpha_0 x_0^* J.$$

Clearly, $R_J(x_0) = \alpha_0$. □

3. The indefinite Rayleigh ratio in Hamiltonian dynamics

The concept of Krein space is encountered in Hamiltonian dynamics. The dynamical state of an n-dimensional Hamiltonian system is characterized by a time dependent vector $v = v(t) \in \mathbb{R}^{2n}$, whose components are the canonical momenta and coordinates, respectively, $p_k, q_k, k = 1, \ldots, n$,

$$v = (p_1, \ldots, p_n, q_1, \ldots, q_n)^T.$$

Denoting the Hamiltonian function by $H = H(p, q) = H(p_1, \ldots, p_n, q_1, \ldots, q_n)$, the time evolution of the components of v is determined by the Hamilton equations,

$$\dot{p}_k = -\frac{\partial H}{\partial q_k}, \quad \dot{q}_k = \frac{\partial H}{\partial p_k}, \quad k = 1, \ldots, n.$$

For physical consistency, the Hamiltonian should be bounded from below, so that it is natural to suppose that it has a minimum at a finite point. Assume that the minimum is attained at the origin ($v = 0$), and so the above partial derivatives vanish at the origin. For small amplitude oscillations the Hamiltonian may be expanded as

$$H(p, q) = H(0, 0) + \frac{1}{2} \left.\frac{\partial^2 H}{\partial p^2}\right|_{p=q=0} p^2 + \left.\frac{\partial^2 H}{\partial p \partial q}\right|_{p=q=0} pq + \frac{1}{2} \left.\frac{\partial^2 H}{\partial q^2}\right|_{p=q=0} q^2 + \cdots$$

Thus, H is a bilinear form with real coefficients in the coordinates and momenta, and so

$$H = \frac{1}{2} \sum_{k,l=1}^{n} (a_{kl} p_k p_l + b_{kl} q_k q_l + 2c_{kl} p_k q_l), a_{kl} = a_{lk}, b_{kl} = b_{lk}.$$

For the $n \times n$ real matrices $A = (a_{kl}), B = (b_{kl}), C = (c_{kl})$, let us consider the Hermitian matrices

$$K = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}, \quad L = -i \begin{pmatrix} 0_n & -I_n \\ I_n & 0_n \end{pmatrix},$$

where O_n denotes the null matrix of size n and $L^2 = I_{2n}$. Then $H = v^T K v$, and the Hamilton equations may be compactly written as

$$iL\dot{v} = Kv$$

It is natural to interpret the Hermitian involutive matrix L as a metric matrix endowing $\mathbb{C}^{2n}$ with a Krein space structure. We associate dynamical states with vectors of this Krein space. The so-called normal modes are associated with an exponential time evolution, i.e., a time evolution given by the exponential factor $\exp(i\omega t)$, where ω is a normal frequency. Normal modes and normal frequencies are the eigenvectors and the eigenvalues of LK, respectively, being determined by the eigenvalue problem

$$\omega L u = K u, u \in \mathbb{C}^{2n}. \tag{4}$$

The Rayleigh ratio

$$R_L(u) = \frac{u^* K u}{u^* L u} = \frac{u^* L(LK) u}{u^* L u}, \quad u \in \mathbb{C}^{2n}$$

is stationary at the normal modes and its minimum for $u \in \mathbb{C}^{2n}$ such that $u^*Lu > 0$, is the frequency of the *fundamental harmonic*. For simplicity we consider the model with $C = 0$, which ensures time-reversal invariance. Under this assumption, it follows that

$$Au = -i\omega v, \quad Bv = i\omega u$$

so that

$$Au = i\omega(-v), \quad B(-v) = -i\omega u.$$

This implies that the eigenvalues of LK in (4) occur in symmetric pairs $\pm\omega_j$. Moreover, the norms of the eigenvectors associated with positive and negative eigenvalues have opposite signs. Since the origin is a minimum, K is positive definite and so the eigenvalues of LK do not interlace. Thus, considering the L-Rayleigh quotient relative to the matrix LK, we conclude, by Theorem 2.1 (I_0), that the set $\{R_L(x) : x^*Lx > 0\}$ is a half-line. Conversely, if $\{R_L(x) : x^*Lx > 0\}$ is a half-line, then, by Theorem 2.1 in [2], the matrix LK does not have complex eigenvalues and, henceforth, dynamical stability is ensured.

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